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ECONOMETRICA
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THE TRANSFORMATION OF VALUE IN THE PRODUCTIVE PROCESS

By L. AMOROSO

I. THE THREE FUNDAMENTAL PRINCIPLES

LET US CONSIDER an economic complex (system), in which certain products (to the number of m) are manufactured, employing certain factors of production (to the number of n).

Let us call:

$z_\mu dt$ = the quantity of the product μ manufactured in the infinitely small interval dt;

p_μ = the price of the product μ at the time t;

$x_{rs}dt$ = the quantity of the factor of production s employed in the manufacture of r, in the interval dt;

q_s = the price[1] of the factor of production s,

i = the rate of interest,

e = the base of the natural logarithms;

$$\mu = 1, 2, \cdots, m;$$

$$r = 1, 2, \cdots, m;$$

$$s = 1, 2, \cdots, n.$$

The summations

$$(1) \qquad Tdt = e^{-it}\sum_{\mu} p_\mu z_\mu dt, \qquad \theta dt = e^{-it}\sum_{r,s} q_s x_{rs} dt$$

then represent respectively the value of the production and the amount of the costs incurred in the interval dt, while the difference

$$(2) \qquad H = T - \theta$$

represents the instantaneous cash balance (difference between receipts and outgo), with the understanding that all the entries of debits and credits will be discounted in the accounts at compound interest from the origin of the time.

In the doctrine of economic equilibrium the productivity of the plant

[1] Considered, wherever it occurs, as the sum of the quota of interest and amortization, in the sense which is explained in my *Principi di economica corporativa*, Bologna, Zanichelli, pp. 136 ff.

is represented by means of a system of *algebraic equations* which connect the quantities produced with the quantities of the single factors employed and which, in the terminology of Pareto, constitute the bonds of the system. In such a representation, quantities produced and quantities employed stay connected only through the values which they assume at the same moment; wherefore there is no possibility of any mathematical representation whatsoever, even indirect, of the cycle of production. To stop this gap, which is a serious one because it refers to one of the fundamental elements of the productive process, I have represented[2] the productivity of the plant by means of a system of equations which connect the velocities with which the single products flow to the velocities and to the accelerations of the single investments: that is, they connect *each one* of the $z_1, z_2, \cdots, z_m$ *to all* the x_{rs} and *to all* of their derivatives (with respect to time) which are here indicated by x'_{rs}. On the hypothesis which is implied in such a representation rests the first fundamental principle, to which I give the name of *differential principle* in this paper.

With these conventions agreed on, let us consider all the production obtained in the time interval which runs from an initial moment t_0 to a final moment t_1. Its value at the origin of the time is the sum of the instantaneous cash balances and therefore:

$$I = \int_{t_0}^{t_1} H \, dt, \tag{3}$$

H being, by reason of the differential principle, a function of the x_{rs} and of the x'_{rs}.

According to a suggestion of my friend, Dr. Michele Rinonapoli, I designate the second fundamental principle by the name of *principle of euphoria*; I express it by saying that from among all the lines that in the same interval of time radiate from a *given* configuration to reach another *given* configuration the individual chooses the one which leads to the maximum value. Fundamentally it is the principle of maximum satisfaction which is universally assumed as the basis of economics and which thus is opportunely defined more precisely.

As the *third principle* I finally assume the hypothesis that the individual operates under conditions of *competition*, accepting the market prices, both for the products and for the factors of production, excluding every manoeuvre arising from a situation of monopoly, total or partial. In this hypothesis all prices—of the products and of the factors of productions—must be assumed as the *data of the problem*, that is, as known functions of the time *independent of the* x_{rs} *and of the* x'_{rs}.

[2] In "La teoria matematica del programma economico," cited later.

II. THE DIFFERENTIAL EQUATIONS OF ECONOMIC MOVEMENT

From the three fundamental principles the differential equations of economic movement are immediately derived.

In fact, by the principle of euphoria, the x_{rs} should be such functions of the time so that, when their values are fixed for $t=t_0$ and $t=t_1$, whatever t_0 and t_1 may be, integral I attains a maximum. By the differential principle, the function H depends upon the x_{rs} and the x'_{rs} whence Euler's equations

$$\frac{\partial H}{\partial x_{rs}} = \frac{d}{dt}\left(\frac{\partial H}{\partial x'_{rs}}\right), \qquad \begin{matrix} r = 1, 2, \cdots, m, \\ s = 1, 2, \cdots, n, \end{matrix} \tag{4}$$

ought to be satisfied.

Substituting for H the expression (2), bearing in mind that by the principle of competition the prices do not depend upon the x_{rs} and the x'_{rs}, (4) assumes the Lagrange form

$$e^{it}\left\{\frac{\partial T}{\partial x_{rs}} - \frac{d}{dt}\left(\frac{\partial T}{\partial x'_{rs}}\right)\right\} = q_s, \qquad \begin{matrix} r = 1, 2, \cdots, m, \\ s = 1, 2, \cdots, n, \end{matrix} \tag{5}$$

and in this form they represent the *principle of the marginal productivities*, which is active in the dynamics of the productive process. The terms

$$-e^{it}\frac{d}{dt}\left(\frac{\partial T}{\partial x'_{rs}}\right)$$

represent in these equations resistances of *inertia* or internal resistances which arise from the financial burdens, from the monetary profits or losses, from complications or facilities of technology and organization, as they are reflected in the productive process by an acceleration or a retardation of investment.[3]

The prices of the factors of production which figure in the second members of (5) appear vice versa as components of *applied forces* or of *the stress*, which, by combining with the forces of inertia, determine the movement at every instant.[4]

[3] Cf. "La teoria matematica del programma economico," in volume, "Cournot" of the *Collana di Ca' Foscari*, Cedam, Padova, 1939 = XVII, summarized in *Rivista Italiana di Scienze Economiche*, Anno X, fasc. II, February, 1939 = XVI.

[4] The analogy had already been established by Pareto, in connection with the formulae of equilibrium, which are indicated below in (6). Here are his words: "Les prix s'introduisent dans les équations de l'économie politique précisement de la même manière que les tensions des liens dans les équations de la mécanique rationelle." Cf. V. Pareto, *Les systèmes socialistes*, Paris, Giard et Brière, 1902, Vol. I, p. 379. See further, V. Pareto, *Cours d'Economie Politique*, Lausanne, F. Rouge, 1896–97, Vol. II, pp. 411–412.

When T is thought to be independent of the x'_{rs}, as is usual in the classical doctrine, the resistances of inertia are wanting, (5) is reduced to the simple form:

$$e^{it}\frac{\partial T}{\partial x_{rs}} = q_s, \qquad \begin{matrix} r = 1, 2, \cdots, m, \\ s = 1, 2, \cdots, n, \end{matrix} \tag{6}$$

which represents the leveling of the marginal productivities, which is active in a stationary regime and which is represented in the theory of economic equilibrium.

Generally, equations (5) constitute a system of mn differential equations of the second order which define the unknown functions x_{rs} except for $2mn$ arbitrary constants, which can be determined:

(a) by prescribing the *configuration of the system* and the *direction of movement* at the starting point; that is, the values which the x_{rs} and the x'_{rs} assume at the initial instant;

(b) by prescribing the configuration of the system at the starting point and at the termination: which is equivalent to stating the values which the x_{rs} assume both at the initial instant and at the final instant.

Conditions (a) represent the behavior of the individual in an evolutionary process directed to obtaining the maximum return with respect to the *existing initial situation*, in which case the goal cannot be prescribed; it will be what it turns out to be. Conditions (b) represent instead a revolutionary process directed toward realizing a *program* or *an economic plan*, tending to modify the existing situation. Not always is the program realizable, because conditions (b) are not always compatible with equations (5). But even when they are compatible, their action implies a *revolutionary process*, because the solutions which are derived from them for x_{rs} tend in general toward a *discontinuity* of the derivatives x'_{rs} at the initial instant.

III. SYNTHESIS OF THE CLASSICAL AND HISTORICAL SCHOOLS

Equations (5) are analogous to Lagrange's equations of classical mechanics, just as the principle of euphoria is analogous to the *principle of least action*. In historical sequence Lagrange's equations precede the principle of least action. They sum up our ideas of the movement of material systems, conceived as the result of the action of applied forces, which had been observed since the classical antiquity of Archimedes; and of the forces of inertia, which were foreseen more than seventeen hundred years later by Galileo. Fundamentally it was the intuition of Galileo that got over the reef on which dynamics had been stopped in the dead angle of Aristotle's deviations. Only later, in a rather advanced phase, was the principle of least action dimly seen (Euler,

Maupertuis, Hamilton), the meaning of which is yet to-day variously interpreted.[5]

The parallel principle of Economics is, on the other hand, immediately evident. It is the *principle of maximum satisfaction*, which is the basis of all economic activity, and from the beginning it was postulated as the foundation of the doctrine. In this article it constitutes the point of departure, from which the equations of economic movement (*analogous to Lagrange's equations*) are derived by supposing that the quantities produced will be functions not only of the quantities invested, *but also of their derivatives with respect to the time.* These equations contain by implication the expression of a *principle of inertia* (analogous to Galileo's principle). It can be expressed by saying that in an economic complex, in which one disregards all considerations of cost and the prices of the products appear as indications of choice among the different possible combinations, the conduct of the individual is determined by the equations:

$$\frac{\partial T}{\partial x_{rs}} = \frac{d}{dt}\left(\frac{\partial T}{\partial x'_{rs}}\right), \qquad \begin{matrix} r = 1, 2, \cdots, m, \\ s = 1, 2, \cdots, n. \end{matrix} \tag{7}$$

The name of *Galileo's system* could be given to an economic system of this kind.

Thus we see that Pareto's system, which sums up the general theory of economic equilibrium in the algebraic representation of a stationary movement, can be extended in the more ample schema of a differential system, which is capable of representing a variable movement and this gives an image of economic life in its alternating fluctuations and in its perpetual progress.

Pareto's is certainly the most pleasing representation of economic life in the spirit of the classical doctrine. But it was universally accepted and before any other by Pareto himself that in addition to Pareto's functions (tastes and the obstacles), there existed in the economic complex other forces and other resistances, which bring about the course of the configuration of equilibrium. Pareto's application to sociology, in the final period of his scientific activity, represents, to express it in military terms, a flanking manoeuvre conceived for the purpose of breaking the siege by which the citadel of economic equilibrium was hemmed in. The manoeuvre had already been initiated many years before in the *form of antitheses* by the historical school. In the maze of the exaggerations, paradoxes, errors, systematic negations with which the reaction of this school is expressed, there is, however, a fundamental truth which is summed up in this assertion: *the economic*

[5] Cf. M. Born, "Cause, Purpose and Economy in Natural Laws," *Proceedings of the Royal Institution of Great Britain*, 1939, pp. 596–628.

movement is the expression of a historical process, in which the present is at every moment conditioned by the past and, in its turn, conditions the future: the dead city is alive in the living city.

The differential representation adjusts itself precisely in taking account of such interferences. By imposing between the several variables a bond rather less rigid than that which resulted from the algebraic link, the doctrine is allowed to adjust itself to represent *the individual in the general*. It represents, in this way, a *synthesis between the mechanical and the historical conception*.

IV. THE POTENTIAL ENERGY OF AN ECONOMIC SYSTEM

An industrial plant is a complex of workers, equipment, and organization, thanks to which certain factors of production—raw materials, labor, service of capital—become changed into one or more products. In the physical aspect the change is a material transformation; in the economic aspect it is a transformation of value.

Is it possible to combine the economic concept of value and the mechanical concept of energy in such a way as to liken the transformation (*of value*) which is affected in the productive process to the transformation (*of energy*) which is displayed in the mechanical process?

In order to answer this question let us return to formulae (6) which represent the equilibrium of the producer in the simple form in which it is conceived in the doctrine of equilibrium. *Let us say here that* $i=0$, let us multiply both members by dx_{rs}, and let us sum with respect to both the indices r and s; we get the formula:

$$\sum_{r,s} \frac{\partial T}{\partial x_{rs}} dx_{rs} = \sum_{r,s} q_s dx_{rs},$$

which can be written

$$dT = d\theta. \tag{8}$$

This expresses synthetically Ricardo's principle of the *leveling of the marginal cost to the value of the marginal production*.

Let us see how this well-known relation can be interpreted as a form of energy.

For this purpose let us observe that for every industrial plant there exists a configuration which corresponds to the maximum of exploitation, in the sense that *no further* investment is possible. That means that, if there are n factors of production, there exist n positive numbers $M_1, M_2, \cdots, M_n$, which represent the maximum limit of the quantity employable for each one. Whence we have always

$$X_s \leqq M_s,$$

where X_s indicates the sum $\sum_r x_{rs}$ and so represents the quantity of s, employed complexly in the plant.

The sum

$$M = \sum_{s=1}^{n} q_s M_s$$

represents, then, the maximum value investable at current prices. Since the quantities actually employed are X_s, the plant has a reserve production capacity which at the current prices is worth

$$V = \sum_s q_s(M_s - X_s) = \sum_{r,s} q_s(M_s - x_{rs}). \tag{9}$$

We see, then, that the quantity V, which in its economic meaning represents a cost and hence a value, may also have the meaning of a *capacity*, that is, of a *potential energy*[6] (of the plant). Its differential, which, *since the prices are by hypothesis stationary*, is expressed by

$$dV = - \sum_{r,s} q_s dx_{rs}, \tag{10}$$

represents a *difference of potential*[7] and coincides *except for the sign* with the marginal cost $d\theta$, so that we have

$$dV = - d\theta. \tag{11}$$

Equation (8) then becomes

$$dT + dV = 0 \tag{12}$$

and tells us that in a fixed configuration production is extended up to the point where the difference of potential dV is equal, with opposite sign, to the differential of the value of the product dT.

Thus we see that to *Ricardo's principle* of marginal cost *may be given an interpretation in terms of energy*, if it is possible to assign to T—value of the production—the character of kinetic energy.

[6] Dr. Michele Rinonapoli, whose advice has been most valuable to me in the development of the present article, calls my attention to a passage of Clark, in which the ideas here indicated have a practical and efficacious illustration. It is the following passage: J. M. Clark, *Studies in the Economics of Overhead Costs*, The University of Chicago Press, Chicago, Illinois, 1923, Chap. IX, §4, pp. 183–186.

The same connection—with reference to the complex of world economics—forms the underlying warp over which is woven the beautiful monograph No. 44 of the Institut für Konjunkturforschung, which is entitled: *Energiequellen der Welt*, Berlin, 1937, Hanseatische Verlagsanstalt, Hamburg.

[7] By formula (10) we find that the prices are the derivatives of the potential function V, with a change of sign. The identification of the prices with the external forces or tensions of a mechanical system thus appears in a new light.

But such an interpretation is not only possible, I should say it is almost natural, inasmuch as in the *model of the classical doctrine*, which we are here considering, *the value of the product is the only index which represents synthetically the result of the productive process*. If the plant works at full capacity, the potential, according to formula (9), is zero and vice versa the product is a maximum and it is natural to say, then, that the kinetic energy is a maximum. If on the contrary the plant is idle, the potential assumes the maximum value M, but the economic activity is zero; and it is equally natural to say that the kinetic energy is zero.

Thus we see that an affirmative reply must be given to the question placed at the beginning of the preceding section.

V. THE LATENT PRODUCTION—EXPRESSION OF KINETIC ENERGY

The study of Ricardo's model has brought us to a definition of potential energy which, except for one variant which will be spoken of later, may be accepted in general. It is not so for kinetic energy, inasmuch as *its identification with the value of current production* is not valid outside the particular model which gave it origin.

The result obtained shows us, however, which is the way to follow in general. The classical model, which has been considered above, is characterized by two hypotheses, which are implied in the doctrine and which we here state with precision:

(a) The productivity of the plant is represented by a system of algebraic equations which connect the quantities of every product to the quantities of every factor employed.

(b) The prices—both of the products and of the factors of production—are assumed to be invariable with respect to the time; the rate of interest is supposed to be zero.

For the purpose of untying the knots one at a time, let us uphold firmly for the moment hypothesis (b) and let us free ourselves of (a), substituting for it the hypothesis expressed by our differential principle.

Following the same clue which has guided us in the classical model, let us start, then, from equation (5); let us multiply both the members by x'_{rs}, and let us sum with respect to both indices r and s. By simple and well-known transformations we arrive, then, at the formula

$$(13)\quad e^{it}\left[\sum_{r,s}\left(\frac{\partial T}{\partial x_{rs}}\,x'_{rs}+\frac{\partial T}{\partial x'_{rs}}\,x''_{rs}\right)-\frac{d}{dt}\sum_{r,s}\left(x'_{rs}\,\frac{\partial T}{\partial x'_{rs}}\right)\right]=\sum_{r,s}q_s x'_{rs},$$

which, after taking into account (10) and the simplifications which hypothesis (b) implies, is written

$$\frac{dT}{dt} - \frac{d}{dt}\left(\sum_{r,s} x'_{rs}\frac{\partial T}{\partial x'_{rs}}\right) + \frac{dV}{dt} = 0 \tag{14}$$

or

$$\frac{dU}{dt} + \frac{dV}{dt} = 0, \tag{15}$$

on the assumption that

$$U = T - \sum_{r,s} x'_{rs}\frac{\partial T}{\partial x'_{rs}}. \tag{16}$$

Since we have identically

$$\frac{\partial U}{\partial x'_{\rho,\sigma}} = -\sum_{r,s} x'_{rs}\frac{\partial^2 T}{\partial x'_{rs}\partial x'_{\sigma\rho}},$$

it follows that, as $x'_{rs}=0$, all the derivatives of U with respect to the x'_{rs} are still null; then U represents the value of the production, such as it would be in an *hypothetical situation*, in which the configuration *would stabilize* its factors of production *actually employed*. To the quantity U thus defined let us give the name of value of the *latent production:* it represents the *kinetic energy* of the system.

In particular, if the x'_{rs} are *identically* zero, the configuration coincides with that represented by Ricardo's model, U coincides with T, the latent product coincides with the current product, and both these quantities are identified with the kinetic energy of the system, just as it was defined above.

In general, (15) gives us

$$U + V = \text{constant},$$

whence we conclude: If the prices are supposed to be invariable and if the effect of interest is neglected, the transformation in value which is effected in the productive cycle may be interpreted in the form of energy in the sense that *the sum of the values expressed by the potential energy of the plant and of the kinetic energy is constant.* This means that *every* variation of cost—up or down—must find its exact equivalent *in the increase or in the decrease of the latent product.*

Formula (16) may even be interpreted as follows: If the prices are supposed constant and if the effect of interest is neglected, the employment of the factors of production—*in the dynamics of the productive process*—is extended up to the point where the *marginal cost is reduced to the value of the latent marginal production.* Particularly, if the cycle of production is assumed to be *instantaneous*, the latent product coincides with the current product and formula (15) comes to be identified with Ricardo's formula.

VI. DISPERSION AND CONCENTRATION OF MONETARY ENERGY

It is easy now to proceed to the general case, by eliminating the restrictions implied in hypothesis (b).

At the base of the definitions of T and of U expressed by formulae (1) and (16), we have identically

$$\frac{dT}{dt} = \sum_{r,s}\left(\frac{\partial T}{\partial x_{rs}}x'_{rs} + \frac{\partial T}{\partial x'_{rs}}x''_{rs}\right) + e^{-it}\sum_{\mu} z_\mu(p'_\mu - ip_\mu),$$

$$\frac{dU}{dt} = \frac{dT}{dt} - \frac{d}{dt}\left(\sum_{r,s} x'_{rs}\frac{\partial T}{\partial x'_{rs}}\right),$$

having indicated by p'_μ the derivative of p_μ with respect to time.

On the other hand, if, for the definition of the potential, which is expressed by (9), we substitute

$$W = e^{-it}V = e^{-it}\sum_{r,s} q_s(M_s - x_{rs}),$$

which coincides with (9) if $i=0$, we have still identically

$$\frac{dW}{dt} = -e^{-it}\sum_{r,s} q_s x'_{rs} + e^{-it}\sum_{r,s}(q'_s - iq_s)(M_s - x_{rs}),$$

having indicated by q'_s the derivative of q_s with respect to t. Substituting in (13) we get by condensing

$$\frac{dU}{dt} + \frac{dW}{dt} + \frac{d\Omega}{dt} = 0, \tag{17}$$

having put

$$d\Omega = \left\{\sum_{\mu} z_\mu(ip_\mu - p'_\mu) + \sum_{r,s}(iq_s - q'_s)(M_s - x_{rs})\right\}e^{-it}dt. \tag{18}$$

The quantity Ω represents a form of *monetary energy*, expressed by the values which are formed by the fluctuations of the prices and by the accumulation of the interest to be paid which is a burden on the circulating capital and on the proportionate part of the fixed capital which temporarily remains idle. It can be compared to thermal energy: the likeness is perfect as for the load of interest to be paid, which always represents a decrease in value and tends therefore to a dispersion of energy; but not for the fluctuations of the prices, which may occur in one direction or in the other and may represent a dispersion or a concentration of energy.

By integrating we obtain from (17)

$$U + W + \Omega = \text{constant}, \tag{19}$$

and this formula sums up—in the most general way—the transformation of value which is effected in the productive process. It tells us that in the dynamics of the productive process:

"*the sum of the values* expressed by the three forms of energy, kinetic, potential, monetary (thermic), *remains constant*,"

or also:

"every *variation of cost* must find its equivalent in a variation of the value of the *latent product*, except the difference due to fluctuations of prices or to the *accumulation* of interest paid";

or finally:

"the employment of the factors of production is extended in every instant up to the point where the marginal cost is brought down to the value of the marginal latent production, to the net amount of the price fluctuations and of the accumulation of interest to be paid."

So we conclude that the *transformation of value which occurs in the dynamics of the productive process can be likened to the transformation which is effected in a mechanical process* and like the latter is governed by a principle analogous to that of the *conservation of energy*, with this fundamental difference: that the conservation of energy in the mechanical process represents a *natural law* which teaches us how certain facts occur, while, on the contrary, the transformation of value which is effected in the productive process represent a *rule of conduct*, which tells us how the facts occur, *if the conduct of the individual is affected by a criterion of rationality*. This criterion of rationality is herein indicated in the *principle of euphoria* which, by dominating the economic process in so far as it is the product of a conscious will, which selects the best means for obtaining its own ends, *explains the reason of the economic action*.

The same cannot be said of the corresponding *principle of least action*, which, not being anything but a felicitous *definition*, leaves the enigma of movement unsolved.

University of Rome

THE ECONOMIC LIFE OF INDUSTRIAL EQUIPMENT

GABRIEL A. D. PREINREICH

WHEN TO REPLACE individual units of durable equipment by similar or improved units is one of the main problems, upon which the success of industrial enterprise depends. Nevertheless, no unified presentation of its many aspects appears to have been published up to the present. The principal writers refer to replacement merely incidentally, when discussing the subject of depreciation. From the theoretical point of view, such an approach really amounts to putting the cart before the horse.[1] Replacement is the basic problem, because it actually affects the composition and productivity of a plant. Calculations of depreciation are mere figures entered into books, the significance of which depends entirely on the use to which they are put. The concept of depreciation does not enter into the theory of capital value at all. In practice, on the other hand, differences in depreciation methods do to some extent influence the judgment of traders in the negotiable symbols of composite capital goods. This anomaly is due partly to defective accounting methods. A study of the replacement problem by itself must precede attempts to correct the situation.

The value aspect of replacement or "economic life" arises from the familiar phenomenon that many types of "machines" outlive their usefulness. The income stream derived from their operation gradually declines, until a more attractive alternative becomes available. The theory that the economic life of a machine is a period which makes the unit cost (plus interest) of the product a minimum, appears to have been originated by Professor J. S. Taylor.[2] His algebraic presentation was simplified and refined by Professor Harold Hotelling,[3] who employs continuous functions for the purpose. The basic formula given by the latter writer is:[4]

[1] I have done that too in "Annual Survey of Economic Theory: The Theory of Depreciation," ECONOMETRICA, Vol. 6, July, 1938, pp. 219–241. The present article is an attempt to organize and expand the comments there made on the replacement problem.

I am greatly indebted to Professors James C. Bonbright, Ragnar Frisch, and Harold Hotelling for the interest they have taken in my MS at various stages of its preparation. Professor Bonbright's unfailing readiness to discuss the main trend of reasoning helped me greatly in clarifying my ideas and their presentation. Instances of mathematical obscurity pointed out by Professor Frisch were corrected in footnotes. Professor Hotelling's extensive comments and my replies will be found in the appendix, which is submitted in lieu of revisions in the text.

[2] "A Statistical Theory of Depreciation," *Journal of the American Statistical Association*, December, 1923, pp. 1010–1023.

[3] "A General Mathematical Theory of Depreciation," *ibid.*, September, 1925, pp. 340–353.

[4] P. 343, formula (2).

$$(1) \qquad B = \int_0^T [wQ(t) - E(t)]e^{-\int_0^t i(\tau)\,d\tau}dt + S(T)e^{-\int_0^T i(\tau)\,d\tau}.$$

In this changed notation, B = original cost of a single machine, T = unknown date at which it ought to be discarded, w = unknown lowest unit cost (plus interest) of the product, $Q(t)$ = rate of production, $E(t)$ = combined rate of all expenses, except depreciation and interest, $i(t)$ = rate of interest, and $S(T)$ = selling price (scrap value) of the machine, when discarded.

By differentiating with respect to T, the unit cost may be written:

$$(2) \qquad w = \frac{E(T) + i(T)S(T) - S'(T)}{Q(T)}.$$

"This equation states that the cost of a unit of product is found by adding the operating cost $E(T)$ of the machine (at the time when it is least efficient and about to be scrapped) to interest $i(T)S(T)$ on the scrap value and the rate of depreciation $-S'(T)$ of the scrap value and dividing this sum by machine's rate of production."[5] The result will be a minimum, when T is determined by substituting (2) in (1) and solving. Professor Taylor's algebraic formula corresponds to equation (1) solved for w, viz.:

$$(3) \qquad w = \frac{B - S(T)e^{-iT} + \int_0^T E(t)e^{-it}dt}{\int_0^T Q(t)e^{-it}dt},$$

from which he obtained the minimum by successive trials equivalent to $dw/dT = 0$. Since the argument is not essentially concerned with the variability of the rate of interest, it is permissible to employ a constant rate for brevity.

Neither of the authors cited defines clearly the exact limitations, within which he considers this method valid. In a general way, both have in mind principally the static situation, where a machine will be replaced by another of identical type, operated under the same economic conditions. With respect to dynamic developments, Professor Taylor merely hints that "if replacement alternatives were changing, whether through changing operating costs or through changed service unit requirements, the optimum economic life of the machine in use would be altered and it would continue in operation for any period for which its discounted operating cost was less than the unit cost plus of the best current replacement alternative."[6] This statement apparently expresses the condition:

[5] Hotelling, p. 345.
[6] Taylor, p. 1022.

$$(4)\qquad w_1 = \frac{E_1(T_1) + iS_1(T_1) - S_1'(T_1)}{Q_1(T_1)} = \frac{E_2(T_2) + iS_2(T_2) - S_2'(T_2)}{Q_2(T_2)} = w_2$$

where the entire economic life T_2 of the replacement alternative must first be calculated from (1) and (2), or (3). The unexpired life T_1 of the machine now in service is then allegedly determined.

Closer examination shows that the problem of economic life is not quite so simple as Professor Taylor's sketch implies. Some of the principal ramifications worthy of study may be classified under three headings:

A. Scope

1. A single machine;
2. A finite chain of replacements;
3. An infinite chain;
4. A number of parallel chains, whose replacement dates are evenly staggered;
5. A large plant continuously renewed in accordance with natural variations in the behavior of similar machines.

B. Limitations

1. Scarcity of new machines available for replacement;
2. Scarcity of various operating facilities or ingredients of production;
3. Scarcity of demand for product;
4. Scarcity of capital;
5. Regulation of profit by law.

C. Economic conditions

1. The static case, where only variations due to the age of the machine are considered;
2. Variations due to the number of co-operating machines;
3. Change in ownership and outlook;
4. Change in the type of machine used (obsolescence);
5. The general dynamic case, embracing extraneous influences as well.

It would be difficult to exhaust the implications of this triple classification. All that can be done is to outline the general trend of reasoning by building up the main problem gradually.

I

When a single machine will not be replaced, it becomes immediately apparent that economic life depends, not on the unit cost, but on the market price of the product. If the capital value of a single-machine enterprise shall be a maximum, the equations to be solved simultaneously are:

$$V = \int_0^T [zQ(t) - E(t)]e^{-it}dt + Se^{-iT} \tag{5}$$

and

$$dV/dT = 0 = zQ(T) - E(T) - iS. \tag{6}$$

This idea is analogous to (1) and (2), except that the unknown capital value V was substituted for the known original cost B and the known market price z of the product for its unknown "unit cost plus" w. The variability of the scrap value is also disregarded here, but only for brevity. The most lucrative life-span T of the machine can now be found from (6) alone and inserted in (5) to find V.

Partisans of the Taylor theory invariably point out that no machine can have a capital value greater than replacement cost. Any goodwill in excess of that amount must be due, not to the machine, but to extraneous and intangible advantages. This argument is valid enough for a number of purposes, but the calculation of economic life is not among them. Since no income whatever can be had without the machine, the entire value of the enterprise must be imputed to it to determine the proper date of scrapping.[7]

A simple example is presented in Figure 1. The data are $z=\$10$, $Q(t)=1-0.04t$, $E(t)=1+0.2t$, $B=\$19.5016$, $i=0.05$, and $S=\$4$. The correct economic life is accordingly $T=14.6\dot{6}$ years, whereas the result from the Taylor-Hotelling calculation would be $w=\$5$ and $T=9.5$ years.

The approach (5), (6) is valid regardless of what the Taylorean unit cost w may be. But replacement is justified only if $z>w$. If the supply of replacements limits the size of the enterprise, a machine will be acquired whenever possible and operated until the date determined by (6). To facilitate reference, I shall call this rule the *individual rule*, since each machine stands only on its own merits.

The situation is altogether different, when only a single machine can be kept in operation at any given time. Starting with a finite chain of replacement, we may write $V=B+G$, where G means goodwill. The capital value of this chain consists of the capital value of the machine

[7] But see appendix, points 3 and 5.

in service, plus the present worth of the aggregate goodwill of all replacements:

$$(7) \qquad G_j = \int_0^{T_j} [zQ(t) - E(t)]e^{-it}dt + (S + G_{j-1})e^{-iT_j} - B.$$

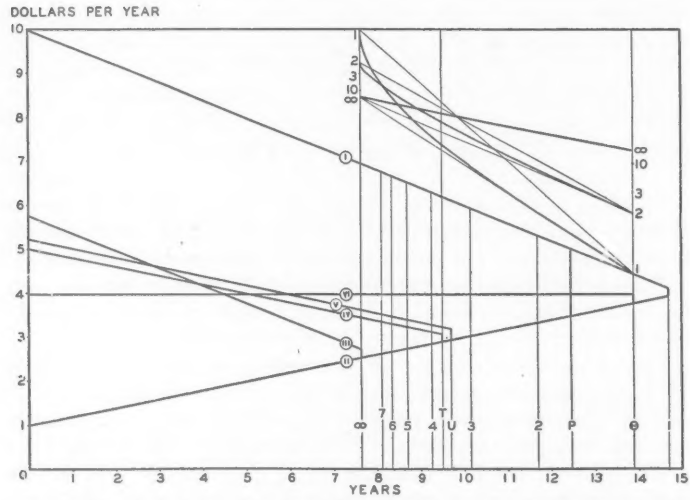

FIGURE 1.—A single machine. Cost $B = \$19.5016$, scrap value $S = \$4$; gross income (sales) (I) $= zQ(t) = \$10(1-0.04t)$; operating expenses (II) $= E(t) = \$(1+0.2t)$.

Economic life: (1) individual rule; (2) · · · (7) · · · (∞) chain rule, depending upon how many successive replacements will occur; (T) Taylor rule; (P) profit rule (discount rate $p_{max.} = 38.31555\%$ per annum); (U) public-utility rule = Taylor rule, when discount rate $p = 7\%$ per annum; (Θ) Taylor rule = chain rule, when sales are constant. Discount rate used for (T) and (Θ) was 5% per annum.

Cost of sales $wQ(t)$: (I) profit rule; (III) infinite chain rule; (IV) Taylor rule; (V) public-utility rule; (VI) Taylor rule = chain rule for constant sales.

Upper right shows auxiliary construction of regression lines of Figure 3. Economic life varies between maximum (Θ) and minimum (∞). Numbering along those ordinates refers to number of evenly staggered parallel chains. See note 11.

The subscript $j = 1, 2, 3, \cdots$, indicates the number of links in the chain. The second relation needed for solution is again the derivative $dG_j/dT_j = 0$, viz.:

$$(8) \qquad G_{j-1} = \frac{zQ(T_j) - E(T_j)}{i} - S.$$

Beginning with $G_0 = 0$, the latter equation determines T_1, the economic life of the last machine in the chain, which is the same as that obtained

from (6). This result, inserted in (7), yields G_1, which is in turn substituted in (8) to find T_2. Returning to (7) we now obtain G_2 and so forth. Each machine will have a longer life than its predecessor and a shorter life than its successor in the chronological chain. The lifespans of the last seven links in a chain of replacements are shown in Figure 1, numbered backward from the end. The data assumed lead to $T_1=14.66$, $T_2=11.687$, $T_3=10.142$, $T_4=9.247$, $T_5=8.696$, $T_6=8.345$, and $T_7=8.116$ years.

As the chain is lengthened, a limit eventually emerges, where $G_j=G_{j-1}$ and $T_j=T_{j-1}$. That is the case of the infinite chain, in which the economic life of all replacements is the same, viz., $T_\infty=7.6461$ years in Figure 1. This answer can be obtained directly by omitting all subscripts and substituting G from (8) for both G's in (7). The equation for the calculation of the standard period T may then be rearranged to read:

$$(9)\qquad z\left[Q(T)-Q(0)-\int_0^T Q'(t)e^{-it}dt\right] = E(T)-E(0)-\int_0^T E'(t)e^{-it}dt - i(B-S).$$

To simplify the expression a bit, the integrations of (7) were performed by parts. Accordingly, $Q'(t)$ and $E'(t)$ represent the rates of change of production and expenses respectively. The criteria of replacement for a single infinite chain and static conditions now become apparent from a generalized graphic solution of (9).

Figure 2 compares the behavior of three types of machines. They are similar in all respects, excepting only their rates of production. The output of the *first* is constant, that of the *second* decreases with growing age (as assumed in Figure 1), and that of the *third* increases with age. Depending upon the functions chosen, the curves will differ in details, but their general behavior is quite uniform. For $T>0$, the left side of equation (9) is always *zero*, when $Q_1'(t)=0$; *negative*, i.e., declining from a maximum of zero, when $Q_2'(t)<0$; and *positive*, i.e., rising from a minimum of zero, when $Q_3'(t)>0$. As for the right side, there is only one curve, rising from the minimum $-i(B-S)$, since the same expense function growing with age serves for all examples.

Upon considering the three types of behavior in turn, we find *first* that, when the rate of production is constant, the market price of the product has no bearing on the date of replacement which is given by the intercept T_1 of the right side of (9) on the axis of abscissae. In such circumstances, formula (9) leads to Taylor's date of replacement.

Curves II and III correspond to the *second* and *third* type and were drawn for the special case, where the product happens to be sold at

its exact minimal unit cost w, as determined by formula (2). The two sides of equation (9) then intersect at points, the abscissae of which could also have been computed by the Taylor-Hotelling method.

When the product is sold at more than its unit cost, i.e., when $z > w$, curves II and III will evidently be deflected toward the axis of ordinates, thereby accelerating replacement for Type II and retarding it for Type III. The two broken curves indicate this development. For

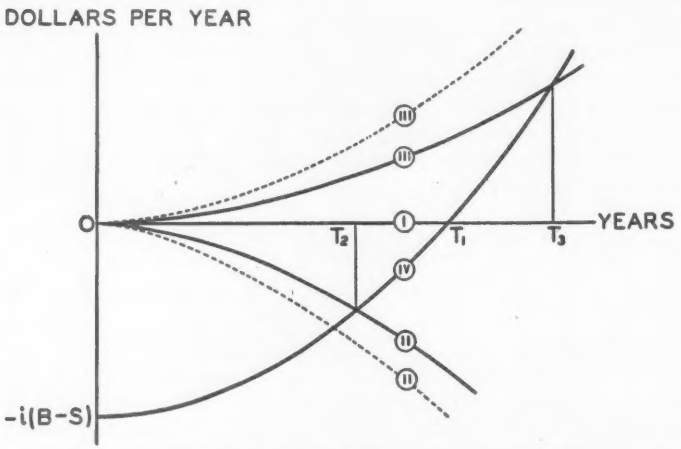

FIGURE 2.—Graphic solution of formula (9). A single infinite chain consisting alternatively of three different types of machines. Type I has constant output; Type II has output declining with age; and Type III has output growing with age. All other characteristics are identical and expressed by curve IV.

Solid curves were drawn for case when output is sold at its exact unit cost $z = w$. Broken curves refer to case $z > w$, showing that increase in profits leaves the economic life of Type I unchanged, but shortens life of Type II and lengthens life of Type III. The graph is not applicable to case $z < w$, which forbids replacement.

the extreme $z = \infty$, Type II would have to be discarded and replaced instantaneously whereas, beyond the limit $zQ_3'(t) = E'(t)$, Type III no longer presents any problem of economic life. Replacement then depends on physical life alone.

Neither formula (9) nor its graph is valid for $z < w$. In such a case, replacement would be uneconomic, but the machine already in service should be operated until the date determined from (6). The lower the selling price, the sooner the enterprise must be abandoned.

The results obtained from (9) can be duplicated by the Taylor technique by pretending that sales above or below a certain constant

volume of output P are deductions from, or additions to operating expenses. The net rental $zQ(t)-E(t)$ may be expressed alternatively as $zP-F(t)$, when $F(t)=E(t)+z[P-Q(t)]$. The economic life T of any machine in an infinite chain is then given by

$$w = \frac{B - Se^{-iT} + \int_0^T F(t)e^{-it}dt}{\frac{P}{i}(1 - e^{-iT})} = \frac{F(T) + iS}{P} \tag{10}$$

regardless of the value assigned to P. This equation is preferable to (9) for general purposes. It seems appropriate to call it the *chain rule.*

Before leaving the single chain of replacements, the complexity of its general dynamic aspect may be outlined. Let T_1 be the unexpired remainder of the economic life of the machine now in service and $T_2, \cdots, T_h, \cdots, T_\omega$ the entire economic lives of the successive replacements. The dates of replacement will then be $D_h=\sum_{j=1}^{h} T_j$, so that equation (7) may be generalized in the form:

$$\begin{aligned} G_{\omega-h+1}(D_{h-1}) = &\int_{D_{h-1}}^{D_h} R_h(t)e^{-\int_{D_{h-1}}^{t} i(\tau)\,d\tau}dt \\ &+ [S_h(D_h) + G_{\omega-h}(D_h)]e^{-\int_{D_{h-1}}^{D_h} i(\tau)\,d\tau} - B_h(D_{h-1}), \end{aligned} \tag{11}$$

where $R_h(t)=z(t)Q_h(t)-E_h(t)$. The index h means that all functions differ not only from machine to machine by virtue of changes in type, but also according to the economic conditions which happen to prevail during the periods T_h, which begin at the unknown dates D_{h-1}.

The derivative of (11) for T_h is:

$$G_{\omega-h}(D_h) = \frac{R_h(D_h) + S_h'(D_h) + G_{\omega-h}'(D_h)}{i(D_h)} - S_h(D_h). \tag{12}$$

Beginning now with $h=\omega$ and $G_0=0$ as in the static case, it is theoretically possible to express from (12) the economic life T_ω of the last machine in terms of known functions of the unknown date $D_{\omega-1}$. The result G_1 from (11) is then substituted in (12), placing $h=\omega-1$ to find $T_{\omega-1}$ in terms of $D_{\omega-2}$. By repeating the process, the aggregate goodwill of all future machines will eventually be expressed as a function of the unexpired life T_1 of the machine now in service. The final step is then to solve (11) and (12) for T_1, when $h=1$, but omitting the last term $B_1(D_0)$ on the right of (11).

This presentation makes allowance for all replacement problems which a so-called "one-horse outfit" can encounter in a dynamic economy. The only requirement is that known functions be substituted for the symbols treated as known. The effect of various degrees

of competition or monopoly, etc., can be readily studied within the framework (11) by using suitable definitions. But if such complications are introduced too soon, the main point may be overlooked. Thus, Dr. C. F. Roos[8] devotes considerable attention to the market price of the product, but considers only a single replacement, i.e., several co-operating chains of only two links each. For longer chains it will not be true, even under static assumptions that "the analogue of the Hotelling hypothesis . . . would require for the replacement problem that the operator endeavor to maximize the sum $V_1(t_1)+V_2(t_1)$."[9] Neither of these two capital values can be found without first knowing those of all subsequent replacements.

From a practical viewpoint, the foregoing demonstration amounts to a *reductio ad absurdum*, for it is obviously impossible to predict how machines, which have not even been invented as yet, will behave under economic conditions prevailing in the dim future. The best that can be done is to estimate somehow the aggregate goodwill of all future machines and then to determine the life of the machine now in service in such a manner that its capital value, plus the present worth of the estimated future goodwill, shall be a maximum. For this purpose it is also reasonable to assume that the goodwill estimate remains unchanged within the limited range of doubt surrounding the date of replacement of the present machine.

To maintain the cost of the product at a variable minimum consistent with dynamic changes is equally impossible, though the underlying theory of public welfare could be readily reviewed by placing $G=0$ and substituting the weighted average unit cost $\overline{w}_{\omega-h+1}$ of all future machines for the market price $z(t)$ in (11) and (12). Returning to Professor Taylor's suggestion, interpreted by equation (4), it may be seen that he considers the unit cost w_2 of the replacement alternative as permanent. In strict theory $\overline{w}_{\omega-1}(D_1)$, an absurdly complicated and untrustworthy function of all future events, would have to be substituted for w_2.

II

Let us now consider the static case, where more than a single chain can be operated, the number being limited only by the demand for the product. For this purpose we may rewrite equation (7) in the form:

$$(13) \qquad G = k\,\frac{\int_0^T [z(\lambda k)Q(t) - E(t)]e^{-it}dt + Se^{-iT} - B}{1 - e^{-iT}}.$$

[8] "A Mathematical Theory of Depreciation and Replacement," *American Journal of Mathematics*, January, 1928, pp. 147–157.

[9] *Ibid.*, p. 156. See also his final formula (6)

The goodwill of k chains equals k times the goodwill of a single chain, regardless of how the chains are staggered as to replacement dates, because the rule found valid for a single chain distributes the stream of excess profits evenly over the life of a machine (see Figure 1). The nature of the demand function (in which λ is the parameter of scarcity) and the method of staggering may introduce summations of successive integrals in lieu of a single integral. In any event, however, the *general rule* consists of applying the theory of maxima and minima with respect to both T and k. To avoid cumbersome formulae, let the numerator of equation (3) be denoted by W [i.e., all terms in the numerator of (13) which are independent of k by $-W$], the denominator of (3) [i.e., the factor of $z(\lambda k)$ in (13)] by Q, and the denominator of (13) by Ii. If derivatives are indicated by primes, the two operations lead to the results:

$$(14)^{10} \qquad dG/dk = 0, \qquad z(\lambda k) + kz'(\lambda, k) = \frac{W}{Q},$$

and

$$(15) \qquad dG/dT = 0, \qquad z(\lambda k) = \frac{WI' - IW'}{QI' - IQ'}.$$

When the division of (15) is performed, subtraction of (15) from (14) gives, upon rearrangement:

$$(16) \qquad \frac{W}{Q} = \frac{W' - kz'(\lambda, k)(QI'/I - Q')}{Q'}.$$

Since the number of machines is a discrete variable, the unadjusted result of differentiation for k implies $k = \infty$ and $\lambda = 0$. On the other hand, $W/Q = W'/Q'$ is the essence of the Taylor method. It follows that this rule is a limit of the general rule, but that it can be valid only when the second term in the numerator vanishes. Apart from the two special cases mentioned in Part I, that can happen only when the market price of the product is independent of the output of an infinitely large number of machines. In addition, all other limitations on the size of the enterprise must be ineffective.

To illustrate this aspect of the Taylor rule, it is necessary to assume that the market can absorb only a given output Pk per unit of time.

[10] Professor Frisch suggested that I clarify the meaning of my symbol $z'(\lambda, k)$ thus: Let $\lambda k = v$. Then, $dz(v)/dv = z'(v) = z'(\lambda k)$, whereas $z'(\lambda, k) = dz(v)/dk = \lambda z'(v)$. The important point is that both $z(\lambda k)$ and $kz'(\lambda, k)$ are functions of the product λk and therefore λ and k occur in the final solution only in the form of that product and not singly.

If so, the optimum number of evenly staggered chains and their renewal period T can be found from:

$$G = \frac{k}{1 - e^{-iT}} \left\{ z \sum_{j=0}^{k-1} \left[\int_0^x P + \int_x^{T/k} Q(t + jT/k) \right] e^{-i(t+jT/k)} dt - \int_0^T E(t) e^{-it} dt + Se^{-iT} - B \right\}. \tag{17}$$

The symbol x represents the recurrent fractional periods, for which the combined productive capacity of evenly staggered machines exceeds the salable output $Pk = \sum_{j=0}^{k-1} Q(x+jT/k)$. Between x and T/k the productive capacity is deficient and sales are lost. To any volume of demand Pk there will correspond a definite number k of parallel chains and a definite replacement period, which makes the aggregate goodwill a maximum. Formula (17), as written, applies to productive capacity declining with age. Should the reverse be true, P and $Q(t+jT/k)$ change places.

The regression lines of Figure 3 are obtainable from the equivalent of (15) by placing $k=1, 2, 3, \cdots$, and varying P in each instance within such limits that x will vary from 0 to T/k. The actual task would be quite tedious, but the principles of a simple graphic solution soon emerge.[11] As the addition of successive chains declines in relative importance, the oscillations of economic life are gradually damped until they merge with the axis $T=9.5$ years, which is the Taylor-Hotelling solution for the data underlying both Figures 1 and 3.

The special expression (13) can now be generalized by letting all functions depend in various ways (e.g., *via* the output) on the product λk, where λ = a common parameter of many different kinds of elastic scarcity. If $dG/dT=0$ and $dG/dk=0$ are then solved simultaneously,[12]

[11] The initial and terminal points of each regression line, the initial tangents on the right, and the levels of indifference are given by the relation of P, k, and x to each other. The final tangents on the left must be vertical. Only the curvatures may be slightly inaccurate, except for the first and first two chains, which I have calculated. The technique is clarified by an auxiliary diagram, in which the ordinates *per chain* of all terminal points are plotted and connected by straight lines. See upper right of Figure 1. For this purpose, ten units of the scale of ordinates represent one unit of annual demand per machine.

[12] Professor Frisch was particularly interested in "the way in which λ and k approach their respective limits," fearing that the results might depend thereon. In the course of correspondence on this point, it developed that my procedure must be explained in greater detail.

In equations (13), (14), and (15), let us assume that k is a continuous variable and that the necessarily discrete number of parallel chains is κ instead. Differentiation with respect to k and T then yields the solution $\lambda k = c$, where λ is a given constant, so that $k = c/\lambda$ will not be an integer, except by accident. If it is

T = axis of convergence and $1/\lambda k$ = distance of successive indifference levels from each other. These levels should be counted from the floor level defined by $G=0$, below which it would not be profitable to replace a machine.

It can thus be shown that the general rule may lead to innumerable different limits, of which the Taylor rule is not the most likely one. To calculate a plausible example would be a task of some magnitude. I have therefore illustrated only a single change in the data of Figure 1, namely $z(\lambda k)=\$10(1-\lambda k)$. As seen in Figure 4, economic life converges rapidly toward an axis of $T=8.42221$ years. A slightly better example of elastic demand would be $z(\lambda k)=\$10\left[1-\lambda\sum_{j=0}^{k-1}Q(t+jT/k)\right]$, but it requires the summation of separate integrals having the successive intervals $jT/k \leqq t \leqq (j+1)T/k$. I have calculated only the limit for $k=\infty$, i.e., the axis $T=11.8324$ years. The Taylor method would give 9.5 years in both instances. The respective goodwills are $\lambda G_{\max.}$ $=0.2487075\times\$79.7042=\$19.8230$ as compared to the Taylorean $\lambda G=0.25\times\$78.3735=\$19.5934$ for Figure 4 and $\lambda G_{\max.}=0.327534$ $\times\$38.6264=\$12.6515$ as against $\lambda G=0.2996193\times\$41.2412=\$12.3566$

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not, I take the two nearest integral values $\kappa=k-a$ and $\kappa+1=k+b$, where $a+b=1$. For the two alternative arguments $\lambda\kappa$ and $\lambda(\kappa+1)$, formula (15) furnishes the two corresponding periods T_a and T_b, one of which is longer and the other shorter than the abscissa of the axis of convergence T derived from the simultaneous equations (14) and (15). Similarly, formula (13) will then have two solutions, viz., G_a for $\lambda\kappa$ and T_a, and G_b for $\lambda(\kappa+1)$ and T_b. Both G_a and G_b must be smaller than the impossible maximum G for λk and the axis T. In the circumstances, the best answer obtainable is either G_a or G_b, whichever be greater. The correct number of parallel chains and the economic life of the machines is accordingly either κ and T_a, or $\kappa+1$ and T_b, as the case may be.

The parameter of elastic scarcity λ in Figure 4 (or the salable output P per machine in Figure 3) is not a variable in this presentation. Changes in the value assigned to it create, not a dynamic problem, but merely so many separate static ones. The value of $\lambda k=c$ is unique for a given set of functions W, Q, and I, so that k is always defined rigidly by λ. In place of k, however, only integral values of κ are available, of which the best must be selected. If the solutions of successive problems are connected by regression lines, Figures 3 and 4 result. To the extreme $\lambda=0$ there will correspond the extreme $k=\kappa+a=\kappa+1-b=\infty$. The difference between a continuous and a discrete variable then disappears and economic life can not differ from the abscissa of the axis of convergence.

The problem could be further clarified graphically in a plane having the coordinates λ and k. A rectangular hyperbola will define all rectangles of equal area $\lambda k=c$. By raising ordinates from the two integral values of κ nearest to any abscissa k, two rectangles are formed, one of which is too large and the other too small. Variation of λ will then lead to a zigzag line oscillating around the hyperbola. But the lower the value assigned to λ, the less the two curves can differ from each other, until both merge at the limit $\lambda=0$, $k=\infty$. The opposite limit of λ is that value Λ, for which $z(\Lambda)=w=$ Taylorean unit cost. Any further infinitesimal increase in λ causes $G<0$ and breaks the last chain of replacements.

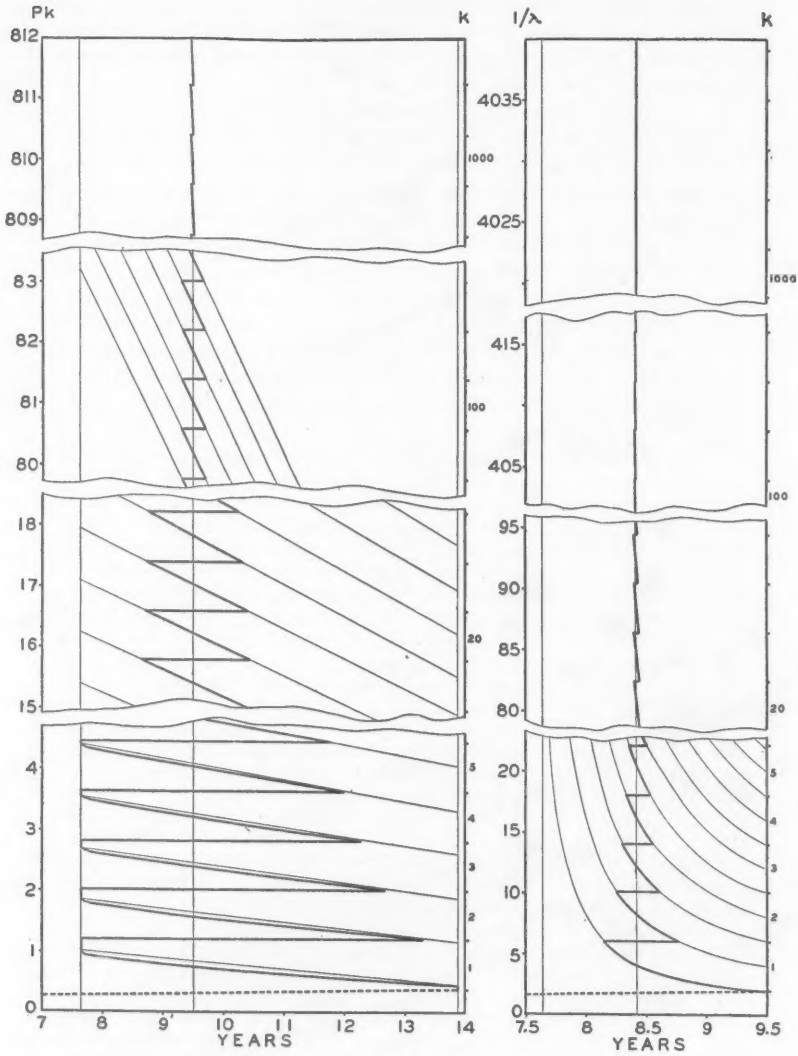

FIGURE 3

FIGURE 4

(For description see page 25.)

for the second example. The coefficients are the various values of λk, when $\lambda=0$ and $k=\infty$. A comparison of the relative magnitudes of infinitely large amounts is thus possible.

The scope of the individual rule must also be mentioned. *First*, it is always valid when $G<0$, i.e., when replacement would be uneconomic. It minimizes the loss on machines already on hand (see broken lines at bottom of Figures 3 and 4). *Second*, it is a limit of the general rule, when a scarcity of renewals is the only effective limitation. This may be shown by restoring z for $z(\lambda k)$ and substituting instead $B(\lambda k)$ for B in formula (13). Operations analogous to (14), (15), (16) furnish the necessary condition $kB'(\lambda, k)=0$. Limited volume at a fixed price, i.e., a rigid scarcity of replacements, must prevail, just as for the Taylor rule with respect to demand. *Third*, the individual rule will affect the problem, when $k\neq\infty$. If the stream of replacements is limited only elastically by rising costs, it is not proper to discard a machine upon acquisition of another. This situation may be stated with slight oversimplifications as follows:

$$G=\frac{k}{1-e^{-iT}}\left\{\sum_{j=0}^{k-1}\left[\int_0^x R(k+1,t+j\theta)+\int_x^\theta R(k,t+j\theta)\right]e^{-i(t+j\theta)}dt + Se^{-iT}-B(\theta)\right\}, \tag{18}$$

where $R(k, t)=z(k)Q(k, t)-E(k, t)$ and $\theta=$interval between consecutive purchases of machines. The loose relationship of k, x, θ, and T to each other can be stated only by stipulating $x=T-k\theta$, when $k\theta<T<(k+1)\theta$. As θ is decreased continuously, k increases by units and recurrent odd periods x are left over, during which there will be one more machine in service. Ultimately $k=\infty$, $\theta=0$, and $x=0$,

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FIGURE 3.—Economic life of a plant consisting of several evenly staggered infinite chains, when the annual demand Pk is rigidly limited in volume, but the market price ($z=\$10$) is independent of that volume. The greater the absorptive capacity of such an artificial market is, the less can economic life differ from the Taylor-Hotelling solution $T=9.5$ years. Machines are of type illustrated in Figure 1. When $Pk=P<0.397756$, the broken line (individual rule) defines the age at which the machine in service, and with it the enterprise, must be scrapped.

FIGURE 4.—Economic life of a plant consisting of several evenly staggered infinite chains, when the market is elastically limited by prices linearly, but inversely related to the number of parallel chains employed. Convergence occurs toward a limit differing from the Taylor-Hotelling solution. Machines are of type illustrated in Figure 1, but market price is $z(\lambda k)=\$10(1-\lambda k)$, where $\frac{1}{2}\geqq\lambda\geqq 0$ is the parameter of scarcity and $k=1, 2, 3, \cdots, \infty$ the number of cooperating chains. When $\lambda>\frac{1}{2}$, i.e., $1/\lambda<2$, the broken line (individual rule) defines the age at which the machine in service, and with it the enterprise, must be scrapped.

whereupon the influence of the individual rule disappears and the simplified approach based on the chain rule becomes correct. Lack of space prevents further consideration of this complication, but its existence should be noted.

III

Up to this point, explicit reference was made only to the first three types of limitation listed in the introduction. Tacitly, however, the principle of the scarcity of capital was also observed in its conventional formulation, according to which additional funds are available only up to the marginal dose, which earns barely the rate of interest. This limitation will now be examined for its own sake.

The capital value of a plant consisting of k machines of all ages from 0 to Θ may be written:

$$Vk = \frac{\int_0^{\Theta/k} \sum_{j=0}^{k-1} R(t + j\Theta/k)e^{-it}dt - (B - S)e^{-i\Theta/k}}{1 - e^{-i\Theta/k}}. \tag{19}$$

When $k=1$, the formula is evidently equivalent to (7) without subscripts. Upon passing to the limit $k=\infty$, we obtain:

$$V\Theta = \frac{\int_0^{\Theta} R(t)dt - b}{i}, \qquad b = B - S. \tag{20}$$

The capital value of a static composite plant equals the present worth of a perpetual income stream of constant intensity. The factor Θ on the left indicates that the number of machines in service is linearly related to the renewal period.

Since the statement is often heard that the owner will do everything in his power to make "enterprise capital value" a maximum, it should be noted that this is true only in a rather technical sense. Literal interpretation would suggest such operations as $d(V\Theta)/d\Theta=0$ or $d[V\Theta/\int_0^{\Theta}Q(t)dt]/d\Theta=0$, which would indeed make capital value a maximum attainable under the corresponding exclusive and rigid conditions of scarcity. Such a procedure, however, would be distinctly detrimental to the owner. The correct attitude is that of a purchaser, who has just acquired the plant in question at a cost C. He will want to know, whether or not there exists a different renewal period T, which would make his purchase more, i.e., most profitable. The search is simplest in the case, where the size of the enterprise is limited only by the stream of replacements available:

$$G = \frac{\int_0^{T} R(t)dt - b}{i} + (\Theta - T)S - C. \tag{21}$$

When the present renewal period is too long, it is best to sell the oldest machines for scrap and replace only the remainder. Differentiation for T leads immediately to the individual rule. Conversely, when the present renewal period is too short, (21) must be rewritten:

$$(22) \qquad G = \frac{\int_0^T R(t)dt - b}{i} - \int_0^{T-\Theta} e^{-it}\left[\int_{\Theta+t}^{T} R(\tau)d\tau + S\right]dt - C.$$

The second term on the right is the capital value of machines between the ages Θ and T, which are nonexistent at the moment, when the lengthening of the life-span is decided upon. Differentiation for T again leads to the same rule. The proof by the Taylor rule is similar in principle, except that the formulae corresponding to (21) and (22) must be so set up as to maintain constant the old composite output $\int_0^{\Theta} Q(t)dt$. This entails a change in both the renewal rate and the number of machines in service.

It can thus be demonstrated in many different ways that "enterprise capital value" must be made a maximum, not for any mature plant, but for one in the process of transition to the optimum renewal period. In terms of the data underlying Figure 1, the greatest capital value obtainable from formula (20) would correspond to $T=15$ years. Nevertheless the economic life under the individual rule is 14.66 because, during the change to the shorter life-span, more capital can be extracted from the plant than the amount by which its capital value will ultimately drop. Similarly, if the Taylor rule should be applicable, $T=9.5$ is a better answer than the maximum at $\Theta=9.072$. The conventional concept of the scarcity of capital is thus illustrated. The conclusion may also be drawn immediately that *economic life is independent of the prices at which the plant is bought and sold.* Whatever the so-called investment C may be, it always disappears in the differentiation process leading to the optimum.

Further scrutiny of the Taylor rule shows that it may also be looked upon as a method of maximizing the gross income per dollar of outgo. By making w from formula (3) a minimum, its reciprocal $1/w$ automatically becomes a maximum. This has also been hailed as proof of universal validity even though, in the general case, it clearly implies a greater scarcity of capital than indicated by the market rate of interest. Under the conventional assumption, it will pay to use additional doses beyond the maximal rate of return, down to the margin.

The "rate of return" maximized by the Taylor rule is an unusual concept, to say the least. When a business man has occasion to use the term, he means something else, namely the annual amount of profit divided by his investment. Operating expenses must be paid out of

current income to keep going, so that he has but little choice in the matter. The need for timely replacement, on the other hand, is less apparent and therefore often underrated. Such neglect enhances the scarcity value or productivity of renewal expenditures beyond those of operating costs. Many plants are in a run-down condition, because the resultant rise in the rate of profit hides the more significant decline in its amount. The limit is reached, when the operator has "spread himself so thin" that he can no longer obtain a loan. A replacement policy then arises, which aims to maximize the rate of return on the investment. The static theoretical solution can be found from any one of the special rules given so far, by substituting the unknown rate of profit p for the rate of interest i. The original cost of a machine always equals the net rental and the scrap value, discounted at the rate of profit. Since the unit selling price of the product accordingly equals the unit cost (plus profit), all approaches lead to the same rule $G(p)=0$, which will be called the *profit rule*. For the data of Figure 1, the answer is $p_{\max.}=38.31555$ per cent per annum (compounded instantaneously) at $T=12.4563$ years.

The profit rule is another limit of the general rule. The substitution of $p_{\max.}$ for i will be theoretically proper, whenever the hire of additional capital would exceed the maximal rate of profit. The owner of several enterprises must equalize the scarcity of capital for all by considering the lowest of the various rates of profit as his private rate of interest. Only the least lucrative enterprise will then be governed by the profit rule; to the rest, the general rule will apply, as determined by all elastic scarcities. As soon as an outsider is willing to lend at a lower, though perhaps still exorbitant rate, that new rate supersedes the former private scarcity rate in all the enterprises owned.

This reasoning leads directly to the correct principle of public-utility regulation. By fixing the "fair rate of return" by law, a rigid condition of scarcity is created, which automatically cancels the influence of all other elastic (or less stringent rigid) scarcities upon economic life. For corroboration, see formula (14), where $G=0$ is a sufficient condition for $kz'(\lambda, k)=0$. It follows that *the profit rule reduces the consumer's costs to a minimum.* Provided only that the fair rate of return be used in lieu of the rate of interest, this rule may be called the Taylor rule, the individual rule, or the chain rule, as preferred.

The theory of public-utility regulation is thus quite definite and leaves no room for equivocation. Most of the difficulties created by dynamic changes could also be overcome by actually fixing, i.e., also guaranteeing the rate of return. A "consumers' surplus (deficit)" account, serving as a temporary reservoir of all differences, would furnish the necessary guidance for "rate" regulation, if its level and rate of

change were currently observed. From an accounting viewpoint, the problem is simple enough, but the clerical cost of calculating all adjustments by hindsight might well be prohibitive. Perhaps sampling methods could be employed.[13] Whether or not it would be sound public policy to guarantee the return is beyond the scope of this article, but the present obstacle race between inflated rate bases and inadequate returns thereon is certainly no solution.

IV

An assumption used throughout the two preceding sections must now be revised. For the purpose of gradual transition from the single-machine problem to that of a large plant, it may be permissible to postulate identical economic lives for co-operating machines of identical type, but observation shows that that is never the case in practice. Some machines drop out very soon, while others continue to render useful service far beyond the average life of a large group.

Statisticians have published voluminous data on the behavior of different types of equipment, but have made no attempt to justify scrapping by any value theory of economic life. They merely assembled the observed facts, namely, how many machines of a given type were actually scrapped at what ages out of what total number.[14] This information, compiled in the form of a histogram, can be normalized and fitted with a frequency distribution $f(t)$, which vanishes for all values of t, except those within the interval $0 \leqq t \leqq n$, where $n =$ maximal age. The frequency distribution cumulated backward is the mortality curve $M(t)$ and the area enclosed by the latter and the co-ordinate axes is the average life a:

$$(23) \quad M(t) = \int_t^n f(\tau)d\tau, \; M(0) = 1, \; M(n) = 0, \; \int_0^n M(t)dt = a.$$

The procedure must now be connected with the value theory of economic life, as outlined in the preceding sections. If the mortality curve is the result of scrapping each machine separately in accordance with the general rule, it follows that the operating expenses and in general also the rate of production differ from machine to machine, despite the identity in type. The symbols of the net rental $R(t) = zQ(t) - E(t)$ should therefore be expanded to $R(y, t) = zQ(y, t) - E(y, t)$, where y expresses the variation from machine to machine. The mor-

[13] For general suggestions, see my articles, "The Principles of Public Utility Depreciation," *Accounting Review*, June, 1938, pp. 149–165 and "The Practice of Depreciation," ECONOMETRICA, Vol. 7, July, 1939, p. 259 and p. 262, point 5.

[14] Cf. Edwin B. Kurtz, *Life Expectancy of Physical Property*, Ronald Press Co., New York, 1930.

tality curve is then defined in terms of $T = M^{-1}(y)$, i.e., by the individual economic lives of many machines acquired at the same time, when those lives are extended horizontally and arrayed from top to bottom, from the shortest to the longest. The inversion $y = M(T)$ discloses, how many per centum survive any age T.

The mechanics of renewal are defined by a variable-limit integral equation of the closed-cycle type, here written in the form of a Volterra equation of the first kind:[15]

$$e^{\int_0^t x(\tau)\,d\tau} = X(t) + \int_0^t u(\tau)X(t-\tau)d\tau, \qquad 0 \leqq t \leqq n. \tag{24}$$

The term on the left is an index of size or volume, when $x(t)$ = rate of growth. On the right, X represents any function of limited variation, which differs from zero only within a range n. The renewal rate is denoted by u. Whenever $n \neq \infty$, equation (24) must be applied to subsequent intervals of n years in the form:

$$e^{\int_0^t x(\tau)\,d\tau} = \int_{t-n}^{nj} u_j(\tau)X(t-\tau)d\tau + \int_{nj}^{t} u_{j+1}(\tau)X(t-\tau)d\tau, \tag{25}$$

$$nj \leqq t \leqq n(j+1).$$

The relationship of renewals to mortality and growth may be expressed in any number of ways, for instance:

I. The *independent renewal rate* u_{I} may be given by the probable future scarcity of new machines available as replacements. Upon placing $X(\alpha) = M(\alpha)$ = mortality under the corresponding limit of the general rule, the right side of (24) and (25) is known and determines an index of the optimum number of machines, which should compose the plant at any given time t.

II. The *number renewal rate* corresponds to situations where it seems easiest to guess the future in terms of the number of machines employable. The left side of (24) and (25) is known and $X(\alpha) = M(\alpha)$ determines u_{II}. This entails a serial calculation for successive values

[15] For the derivation and solution see *op. cit.* in note 1, pp. 221 *et seq.* Since publication of that paper, my attention was called to A. J. Lotka's numerous articles dealing in the main with similar population problems, but including also an excursion into "Industrial Replacement" (*Skandinavisk Aktuarietidskrift*, 1933, pp. 51–63). He employs a technique developed by Paul Hertz in "Die Bewegung eines Elektrons" (*Mathematische Annalen*, 1908, pp. 84–86). This approach consists of substituting a generalized Fourier series for the real solution and gives very poor results during the early years to which foresight can possibly extend. For a demonstration see my paper, "The Theory of Industrial Replacement," *Skandinavisk Aktuarietidskrift*, 1939, pp. 1–9.

of j, so that $u_{\mathrm{II},j}(t)$ is always known from the preceding step, whereas $u_{\mathrm{II},j+1}(t)$ is the function sought.

III. The *output renewal rate* will be obtained, if it is found preferable to guess $x(t)=$future rate of growth in output. In that case, $X(\alpha)=\int_0^{M(\alpha)}Q(y,\alpha)dy$ and equations (24) and (25) must be solved for u_{III} as before.

IV. The *capital renewal rate* is suggested by the profit rule. The left side of (24) and (25) expresses the capital available at various times and therefore on the right $X(\alpha)=\int_\alpha^n e^{-p\nu}\left[\int_0^{M(\nu)}R(y,\nu)dy+Sf(\nu)\right]d\nu$. The task of solving for u_{IV} must again be undertaken.

Carrying theory to extremes, it might be held that the choice of the renewal rate is governed by that one of the innumerable composite limits of the general rule, which happens to be in force in a particular case. Obviously, however, the capital value of the plant must be the same for any kind of renewal rate, if the same rule of economic life was employed in its calculation. Only the unit of measurement will differ, in terms of which the capital value is expressed. Such units are for instance the intensity of the flow of new machines at $t=0$, the originally installed number of machines, the original volume of output, or the original capital. All rates of renewal and all rates of growth are thus convertible into terms of one another.

To find the capital value of the plant, the net rentals of all machines composing it at various times are first added up. This involves two steps, namely summation for each separate age group and totalling for all such groups:

$$r(t)=\int_0^{M(t)}R(y,t)dy+\int_0^t u(\tau)\int_0^{M(t-\tau)}R(y,t-\tau)dyd\tau, \qquad 0\leqq t\leqq n. \tag{26}$$

The first term on the right refers to the machines originally installed and hence vanishes when $t>n$. The second term should then be subdivided as in (25). Purely for brevity of notation, however, it is permissible to designate the renewal rate by $u(t)$ for the entire interval $0\leqq t\leqq\infty$. We may accordingly write:

$$r(t)=\int_{t-n}^t u(\tau)\int_0^{M(t-\tau)}R(y,t-\tau)dyd\tau, \quad n\leqq t\leqq\infty. \tag{27}$$

For a constant rate of growth $x(t)=x\gtreqless 0$, the rate of renewal converges more or less rapidly toward its asymptote:

$$u(\infty)e^{-x\infty}=\frac{1}{\int_0^n X(\tau)e^{-x\tau}d\tau}=U, \tag{28}$$

which can be readily derived from formula (25). The ultimate net rental function then becomes:

$$
\begin{aligned}
r(\infty)e^{-x\infty} &= U\int_0^n e^{-x\tau}\int_0^{M(\tau)} R(y,\tau)dyd\tau \\
&= U\int_0^1\int_0^T R(y,\tau)e^{-x\tau}d\tau dy, \qquad T = M^{-1}(y),
\end{aligned} \tag{29}
$$

and leads to the capital value of a mature plant:

$$
V(\zeta) = \frac{U_1 e^{x\zeta}}{i-x}\left[\int_0^1\int_0^T R(y,\tau)e^{-x\tau}d\tau dy - b - \frac{Sx}{U_2}\right], \tag{30}
$$

$$
i > x, \quad \zeta = t - \infty \geqq 0.
$$

In this formula, U_1 = asymptote of any kind of renewal rate, whereas U_2 = asymptote of number renewal rate, since the scrap values sunk into a growing plant can be related to no other. No similar distinction need be drawn for the rates of growth, as they will all be equal in the end. The formula differs from (20) only by the introduction of individual characteristics and a rate of growth. If those two refinements are deleted, only the constant factor U_1 remains to denote the unit of measurement. The integration with respect to y is merely an averaging process, so that *all conclusions drawn in Sections II and III hold good in terms of the average machine.* The rate of growth does not influence that average because all renewal lots, though unlike in size, are considered large enough to show a similar distribution of individual characteristics.

Let us now consider the problem of obsolescence. With the passage of time, we shall reach the successive dates D_h (where $h = 1, 2, 3, \cdots, \omega$), when replacement by an improved type of machine becomes possible. To simplify the final formula, let us also include immediately the items of income and outgo arising in addition to the net rental. These are the renewal rate and the rate of scrap sales. For machines of type h, we thus obtain the triple expression:

$$
\begin{aligned}
P_h(t) &= \int_{D_{h-1}}^{t} u_h(\tau)\left[\int_0^{M_h(t-\tau)} R_h(y, t-\tau)dy + S_h f_h(t-\tau)\right]d\tau - B_h u_h(t), \\
&\qquad D_{h-1} \leqq t \leqq D_{h-1} + n_{h-1}, \\
&= \int_{t-n_h}^{t} u_h(\tau)\left[\int_0^{M_h(t-\tau)} R_h(y, t-\tau)dy + S_h f_h(t-\tau)\right]d\tau - B_h u_h(t), \\
&\qquad D_{h-1} + n_{h-1} \leqq t \leqq D_h, \\
&= \int_{t-\nu_h}^{D_h} u_h(\tau)\left[\int_0^{\mu_h(t-\tau)} R_h(y, t-\tau)dy + S_h\phi_h(t-\tau)\right]d\tau, \\
&\qquad D_h \leqq t \leqq D_h + \nu_h.
\end{aligned} \tag{31}
$$

How the first of these equations must be amended for $h=1$, may be seen by providing all component elements of (26) with that subscript. Equation (27) is equivalent to the corresponding term of the second form of $P_1(t)$.

At the dates D_h, the replacement of type h stops and the installation of type $h+1$ begins. During the last of the three intervals (31), therefore, the flow of net income is $P_h(t)+P_{h+1}(t)$, where the latter function takes its first form. It should also be noted that although, in the absence of a change in type, forecasts of R_h, B_h, and S_h determine M_h by means of the governing limit of the general rule of economic life, the same relations do not hold with respect to the third variety of $P_h(t)$. The final mortality curve μ_h, its derivative $-\phi_h$, and $\nu_h=\mu_h^{-1}(0)$ are forms of an unknown function to be determined in such a manner that the capital value

$$V(D_h) = \int_{D_h}^{H} P(\tau)e^{i(D_h-\tau)}d\tau + L(H)e^{i(D_h-H)} \tag{32}$$

shall become a maximum. In this formula, $P(\tau)$ represents the partly overlapping series $P_h(\tau)$ for all consecutive values of h. The other new symbols are $H=D_\omega$=horizon, i.e., limit beyond which it is not expected that operations will continue,[16] and $L(H)$=liquidating value, best defined as the capital value after abandonment of further renewals.

The determination of the number renewal rate u_{h+1} may vary according to circumstances. If perfect co-operation between machines of types h and $h+1$ is possible, its first segment will be defined by the equation

$$e^{\int_0^t x(\tau)\,d\tau} = \int_{t-\nu_h}^{D_h} u_h(\tau)\mu_h(t-\tau)d\tau + \int_{D_h}^{t} u_{h+1}(\tau)M_{h+1}(t-\tau)d\tau, \qquad D_h \leqq t \leqq D_h + \nu_h, \tag{33}$$

which contains the same two unknown functions as the corresponding segment of $P(\tau)$ in (32). Multiplication of u_{h+1} by an appropriate constant will be in order, when old and new machines are not exchangeable

[16] Whenever the rate of growth equals or exceeds the rate of interest, the counterpart of formula (30) leads to an infinite capital value. This impossible result calls attention to the element of risk, usually expressed by the addition of an insurance premium to the pure money rate. In my opinion, such a two-in-one tool of analysis is not always flexible enough and therefore I have elsewhere suggested the use of a horizon as a measure of risk in the valuation of common stocks. Cf. my book *The Nature of Dividends*, New York, 1935, p. 10 and mathematical appendix, where two separate horizons are employed to approximate the risks of perpetual earnings and expansion respectively. For present purposes a single horizon will do, so long as the rate of growth is considered variable.

at par. It is also possible, however, that the whole plant of type h must be scrapped immediately,[17] because the two types can not be used side by side. In that case, u_{h+1} is defined by M_{h+1} alone. Many other cases may arise between these two extremes. Depending upon the relationship between the average age of the machines composing the plant at the time D_h and the net proceeds from their forced sale, it may or may not be worth while to operate the abandoned type h for a short time and to curtail the early purchases of type $h+1$ accordingly. The special problem of economic life during a process of liquidation will also emerge at the final date H. The details of (32) will thus vary according to the specific case under consideration. Ordinarily, it is reasonable to assume that only one change in type can be foreseen at any given time, so that the index h can have only the value $\omega-1$. If that should not be the case, solution must begin at the end of the composite chain and proceed backward in a manner reminiscent of single-chain dynamics.

Although this description of the composite chain considers only obsolescence, the general dynamic situation can also be readily visualized. The subscript h identifies not only the type of replacements, but also the successive periods, during which one type is preferred to all others. Its meaning may therefore be expanded to include a reference to all extraneous changes in economic conditions as well. All functions then obtain an additional argument to denote that they are also subject to change with the passage of time.[18]

The index h can also be used to express successive changes in ownership and outlook. For otherwise static conditions, $R_h=R_{h+1}$, $B_h=B_{h+1}$, and $S_h=S_{h+1}$, but $M_h\neq M_{h+1}$ and $u_h\neq u_{h+1}$, when the new owner can improve upon the replacement policy followed by his predecessor. In that case M_{h+1} is given by the new forecast of future conditions, including the nature of the governing rule, while, within the period of transition, M_h is once more an unknown function to be determined so as to make $V(D_h)$ a maximum. This generalizes the simpler discussion based on formulae (20), (21), and (22).

The main structure of replacement theory may be completed by

[17] This problem is discussed in simpler terms by Professor P. O. Pedersen, "On the Depreciation of Public Utilities," *Ingeniørvidenskabelige Skrifter B Nr. 12*, Dansk Ingeniørforening, Copenhagen, 1934, pp. 69–99.

[18] Such a presentation is of course subject to Professor Hotelling's observation that it considers "time as a passive parameter, carrying along the gradually changing influences of a mass of unspecified sources of variation" (cf. "The Work of Henry Schultz," ECONOMETRICA, Vol. 7, April, 1939, p. 99). For purposes of a general theory of economic life, however, it is sufficient to note that, if specifications are available, they can be readily utilized within the framework of formula (32).

mentioning the *liquidating rule* derived from (32) by the operation $dV(t)/dH=0$. This rule is analogous to the individual rule (6) and states that liquidation should take place, when the difference between income and outgo drops below the rate of interest on the liquidating or break-up value of the enterprise. This value is of course the present worth, at the date H, of all proceeds collected, less expenses incurred thereafter. As in (6), the amount of investment shown by the books has no bearing on the optimum.

V

To summarize the conclusions reached, it may be said that the theory of economic life is essentially a theory of scarcity. Successful enterprise has many tangible and intangible ingredients, each of which may be limited either rigidly as to volume, or elastically by price movements. Starting with a total lack of elasticity, it is evident that the relatively greatest scarcity alone determines the size of the plant. All others pass unnoticed. As shown, Professor Taylor's theory presupposes that a rigid demand for the product is the least abundant ingredient. The individual rule and the profit rule are counterparts with respect to the supply of new machines and capital. Analogous rules can be readily developed for rent, labor, fuel, etc.

The general rule of replacement, which is simply the theory of maxima and minima, has a separate solution for every kind of rigid scarcity and for every volume of the supply so limited. When the volume required by a single machine becomes insignificant in comparison to the total, the problem is simplified into making the excess profit (goodwill) per unit of that ingredient a maximum. In the case of demand, that means making the cost per unit of demand (output) a minimum. In all other instances, the limitation operates at the other end of the productive process and therefore the first description applies. The excess profit per new machine, per square foot of space, per hour of labor, per ton of fuel, etc., must be made a maximum, depending upon where the shortage is felt.

Elastic scarcities introduced singly lead to another set of rules. Their relationship to the Taylor-Hotelling theory is best disclosed by copying the Taylor technique exemplified by formula (3). A glance at equation (16) shows that the correct *demand rule* may be written in the abbreviated notation:

$$(34) \qquad w = \frac{W}{Q}\, e^{-ks'(\lambda,k)\int (Q'/Q)dT - Q'/W)dT}, \qquad dw/dT = 0.$$

This formula can also be used to express the limit of the general rule for any other case of isolated elastic scarcity, for instance a *labor rule*,

a *tax rule*, etc. Only the meaning of the symbols must be appropriately generalized. In the numerator of formula (13), let some term other than the sales be dependent on the number of machines and let that term (or subterm) be rewritten in the form "price times rate of consumption." Then let $-W$ = all terms independent of k, Q = discounted volume of the ingredient consumed during the life of the machine, and $z(\lambda k)$ = purchase price per unit of that ingredient. It follows that the counterpart of the Taylorean "unit cost plus" w is that price of the only scarce ingredient, which would cause the goodwill to vanish. By maximizing it with respect to T, the difference $w - z(\lambda k)$ = excess profit per unit of the ingredient will also become a maximum. All operations are the same as for the demand rule, only the terminology and the signs are reversed.[19]

The last step in analysis consists of combining all scarcities. Since all are ordinarily elastic, none may be neglected. Some are no doubt negative, in which case the law of increasing returns must be offset against the law of diminishing returns. The theoretical assumption $k = \infty$ implies that the plant has reached a size, where the latter predominates. The final conclusion is therefore that *excess profits must be made a maximum in terms of a composite index of productive activity, not with reference to any single ingredient*, such as demand.

If all prices, whether paid or received, are denoted by $z_j(\lambda k)$ or z_j for short and all discounted volumes consumed or produced by $\int_0^T Q_j(t)e^{-it}dt$ or Q_j for short, the static replacement problem can be summarized by the symbolic formulae:

$$(35) \qquad \begin{cases} dG/dk = 0 = \sum_{j=1}^{\infty} (z_j + z_j')Q_j, \\ dG/dT = 0 = \sum_{j=1}^{\infty} (IQ_j' - Q_jI')z_j, \end{cases}$$

where $z_j' = kdz(\lambda k)/dk$, $Q_j' = dQ_j/dT$, $I = (1 - e^{-iT})/i$, and $I' = e^{-iT}$. Prices received are positive and prices paid are negative. This notation covers scrap sales and renewal costs also, if $Q_S = e^{-iT}$, $z_S = S$, $Q_B = 1$, and $z_B = -B$. For general dynamic conditions the twin formulae (35) will fail and a guessing process idealized by (32) must take their place.

[19] It is worth repeating that all such special rules can be valid only for $k = \infty$. The expression $kz'(\lambda, k)$ does not thereby become infinite, since it also contains the parameter of scarcity λ, the limit of which is zero. Neither should it be overlooked that the presentation in the form (34) serves only for the calculation of the unit costs. To find the limits of T and λk, equations analogous to (14) and (15) must be solved simultaneously, using the same generalized definitions as in (34).

All rules of economic life are also rules of depreciation, since each suggests the apparently most logical way (out of innumerable other possibilities conforming to the terminal condition) in which costs ought to be distributed in the corresponding circumstances. For instance, if the fuel supply were rigidly limited, all costs should be written off in proportion to the fuel consumption. To exaggerate for the sake of emphasis, it might then be said that the investment consists entirely of fuel. Similarly, it may consist of anything else. In general, it contains a bit of everything. The original cost of a machine could thus be analyzed into its components:

$$B(0) = \sum_{j=1}^{\infty} w_j Q_j, \tag{36}$$

subject to the proviso $j \neq B$, meaning that the term $w_B Q_B = -B(0)$ has been removed from the summation and transferred to the left side of the equation.

In addition to an infinite number of different sets of unit costs varying with the age of the machine, which would satisfy (36), if the w_j had been left under the integral signs included in Q_j, we can also find an infinite number of satisfactory sets of constant unit costs. The most logical of all these sets is that which can be had by considering each ingredient in turn as the only scarce one, i.e., performing all calculations (34), after the limit of λk has been determined and after w_j is substituted for the corresponding prices z_j, which are included in the cumulative symbol W. Each unit cost will thus be expressed in terms of all the others. Simultaneous solution of all equations (34) is then in order. The number of conditions so given is sufficient, since $w_B = z_B = -B(0)$ for this particular purpose.

If that be considered the correct procedure, the theoretical depreciation method for a single machine operated by an unregulated enterprise would be:

$$B(t) = \sum_{j=1}^{\infty} \int_{t}^{T} w_j Q_j(\tau) e^{i(t-\tau)} d\tau, \qquad j \neq B. \tag{37}$$

This expression includes the scrap value, if $w_S Q_S(\tau)$ is defined as $iS/(e^{i(T-t)}-1)$. The only conclusion which I shall draw from this analysis for the present is that it explains, why "unexpired cost" or "investment" must remain a vague and nebulous concept for practical purposes.

Accountants are beginning to realize, how little meaning can be ascribed to balance-sheet figures certified to "conform to generally accepted accounting principles consistently applied in the past."[20] In

[20] The phrase officially adopted by the American Institute of Accountants reads: " . . . conform to generally accepted accounting principles applied on a

view of the extreme complexity of the situation and the difficulty of obtaining the data required, not much can be done beyond facing the facts frankly and striving at least for the comparability of successively reported profits by ironing out, as far as possible, all those short-term fluctuations which tend to becloud the main issue, i.e., the problem of appraising the capital value of an enterprise.[21]

Although I believe I have fairly stated the serious theoretical limitations of the Taylor-Hotelling idea of minimizing unit costs, I nevertheless consider it the most valuable single rule of thumb which can be laid down for the general guidance of the entrepreneur, at least when the number of machines employed is very large and no radical change in type is imminent. Demand is usually the most important ingredient of successful enterprise. In addition, equipment can seldom be exploited to the full extent of its capacity. In some fields, this limit can never even be approached. As far as the machines themselves are concerned, there may thus be no reason why the output of any one should not be approximately constant for successive accounting periods. Extraneous economic conditions will, of course, introduce fluctuations in output and price or, what is more to the point, in the product of both per machine. Even then, however, the trend per machine may not differ substantially from a horizontal line. In such cases, it follows that the natural instinct to minimize unit costs can lead to no great error. In any event, the rule will serve as a useful median or point of departure for guessing the average life under moderately dynamic conditions. In what direction it is apt to err in various circumstances, I have attempted to point out at least in a general way.[22]

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basis consistent with the preceding year" (*Journal of Accountancy*, July, 1939, p. 18). This amendment robs the "certificate," "report," or "opinion" of even the last vestige of its significance. See note 21 below.

[21] For a discussion of the practical criteria governing the choice of a depreciation method see "The Practice of Depreciation," cited in note 13 above. Apart from the conclusions there reached, *long-range* consistency is the most important prerequisite of comparability. Which of the various contradictory, but nevertheless "generally accepted" methods is consistently applied, matters less in the long run. See *op. cit.* in note 1, conclusions 2 and 4 on p. 232. Also p. 240, *ibid.*

[22] These comments refer to unregulated enterprise only. In addition, the Taylor-Hotelling theory is entirely correct from the viewpoint of public welfare, at least so long as each regulated enterprise must still stand on its own feet. Neither of the two authors had considered this aspect at the time the papers cited were written. Since then, Professor Hotelling has dealt extensively with "The General Welfare in Relation to Problems of Taxation and of Railway and Utility Rates" (ECONOMETRICA, Vol. 6, July, 1938, pp. 242–269), but his pertinent conclusion is that "nothing could be more absurd than the current theory that the overhead costs of an industry must be met out of the sale of its products

The practical significance of this résumé is unfortunately impaired by the difficulty of determining the extent to which any particular machine happens to differ from the average of its type. I know of no instance, where it is considered feasible to keep individual operating records in sufficient detail to be of any real help. In addition, unforeseen accidents will happen so that, in the last analysis, physical inspection will probably remain the deciding factor. On the whole then, I am not greatly impressed by the practical merits of the theory of economic life, although it is no doubt a fascinating subject, worthy of study for the sake of its legitimate place in economics. From any other viewpoint, it seems to share the well-known peculiarity of the weather: A great deal may be said, but very little can be done about it!

New York, N.Y.

APPENDIX

In recognition of Professor Harold Hotelling's priority in this field, the MS of this paper was submitted to him for criticism. The resultant exchange of thoughts is here summarized with his permission and in compliance with Professor Frisch's suggestion. The numbered remarks are his, the answers mine.

1. The idea of an infinite chain of replacements which you have developed, seems to me well worthy of consideration. It is interesting that we have in the various dates of replacement an infinite number of variables, with respect to which the discounted profit is to be a maximum. Under static economic conditions, or under economic conditions varying in a preassigned fashion, the infinite chain seems to be merely another way of looking at the problem to which I gave chief attention in my 1925 paper and to yield the same numerical results.

Answer: I agree that the numerical results will be the same, if "static economic conditions or economic conditions varying in a preassigned fashion" are suitably defined in one of three alternative ways, viz., when: (1) so-called perfect competition prevails, i.e., cost and market price can not differ; (2) the rate of sales $z(t)Q(t)$ is constant; and (3) the selling price is independent of the output of a very large (infinite) number of machines and no other limitations exist, which would make the employment of a smaller (infinite or finite) number more profitable.

2. I can see now how it is that you can in some cases arrive at a valuation of a machine that is higher than the cost. Such cases seem

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or services, in order to find out whether the creation of that industry was a wise social policy" (p. 268). I don't object, but it seems to me that, pending the general adoption of such a social philosophy, the public welfare could be greatly enhanced by considering Professor Hotelling's earlier theory as a standard of public-utility regulation.

to call for a limitation in the supply of renewals. In that case, the existing machines will have a scarcity value which might enhance the value above cost. This limitation, however, violates the assumption of static economic conditions, which was made explicitly in the relevant part of my paper.

Answer: That is the most elementary case, used for introductory purposes. The general situation, where all scarce ingredients of enterprise contribute toward the goodwill, is reviewed in Part V. As for the underlying assumptions, a mutually satisfactory agreement could perhaps be reached by saying that the Taylor-Hotelling theory is two-dimensional, i.e., limited to a time-value plane. Within that plane, it is correct, when "static economic conditions" are defined as in the answer to comment 1 above. I have added a third dimension, namely the number of co-operating machines. In the circumstances, the two-dimensional analysis should turn out to be a special case of the three-dimensional one. As may be seen from the text, that is indeed the case.

3. The most vulnerable part of this paper is, I think, the third paragraph on page 15. As to its first sentence, I think no one would deny that the capital value of a machine might be greater than its cost new, provided economic conditions are changing sufficiently. As to the last sentence of this paragraph: "Since no income whatever can be had without the machine, the entire value of the enterprise must be imputed to it, to determine the proper date of scrapping," the argument might be applied to declare that each essential part of an enterprise has a value equal to the whole. Many fallacies of this type are cited by Bonbright in his treatise on value.

Answer: Taken out of its context, this passage does sound a bit vulnerable, although it is correct enough in the simple introductory case, to which it refers, i.e., where the enterprise consists only of a single machine (which will not be replaced) and an intangible differential advantage. When building up a complex problem gradually, it is sometimes hard to avoid oversimplifications at the start. Moreover, the difficulty is a purely verbal one. Whether or not it is proper to refer to the sum of the machine's cost and the goodwill (arising from the co-operation of all ingredients) as the capital value of the machine, my point is that the goodwill must not be excluded from the formula, from which the economic life of the machine is determined. In the three special cases, however, the goodwill drops out, either by definition, or in the process of solution.

4. In distinguishing between the market price z and the value w of a unit of product or, as I call it in my 1925 paper, "theoretical selling price," you may not have observed that in that paper I treated the case in which the market price of the product of the machine was given by

conditions beyond the control of the owner of the machine. As I pointed out near the beginning of the paper, this is a case of somewhat limited applicability, in which the entrepreneur might find himself "resting precariously on the judgment of his competitors." It seemed to me that a more generally important problem was that connected with cost determination in conjunction with the attempt to minimize cost or maximize profit. It was in connection with this more general case that I used the ideas of unit cost and unit cost plus, which had existed for a long time, but had been applied in a slightly inexact manner to this problem by Taylor.

Answer: I recall the passage, but am inclined to hold that the entrepreneur is always resting more or less precariously on the judgments of the consumer, the competitor, and his own. That seems to me the essence of competition. He will always sell above cost, when he can, and temporarily below cost, when he would lose more by closing his plant altogether. But if he does either of these things, the Taylor-Hotelling rule of replacement will no longer be valid, except in the two remaining special cases, both of which assume $z > w$. The attempt to maximize profit is in order, but greater generality will result from applying it to all ingredients of enterprise, not only to demand. The scarcity analysis in terms of the number of co-operating machines is essential for this purpose.

5. In many cases the market price of the product of the machine has no definite meaning because nothing is sold which is the product of one machine alone; the articles sold are the product of many machines under the same ownership, each of which is essential to the finished product. The price of the finished product may well contain an element of monopoly profit or rent in addition to special advantages which can not be assigned unambiguously to any particular physical property. This is the typical situation in industry. In such cases we cannot speak of the value of the product of a machine as determined by external market conditions alone. The "theoretical selling price" used by Taylor and myself becomes a practical tool in connection with cost accounting which should have considerable practical utility under these conditions. Value must be assigned to the service of a machine by the owner of a complicated industrial plant on the basis of the best possible alternative to that service. Under static economic and technological conditions, the best alternative to a machine is typically another machine of the same kind.

Answer: My omission of the typical case, where a product must pass through many plant departments, before being finished, is no doubt a major defect. The correction is not difficult, however. The problem consists of prorating the total selling price Z among the plant de-

partments in such a manner that the various economic lives thereby determined shall make the aggregate goodwill a maximum. If j be the serial number of the departments, we have the unknowns G, T_j, k_j, and z_j, which can be found from an equal number of simultaneous equations $G=\sum_{j=1}^{\omega} G_j$, $dG/dT_j=0$, $dG/dk_j=0$, $z_\omega=Z-\sum_{j=1}^{\omega-1} z_j$, and $dG/dz_{j-1}=0$.

Many intangibles ranging all the way from a patent to advertising slogans behave just like machines in the sense that they not only entail original costs, but are also subject to expiration, deterioration, growing costs of maintenance, etc., and must therefore be renewed from time to time. For present purposes, they may be considered as so many additional plant departments and treated in the same manner. The goodwill is only that residual intangible, which has none of the attributes of a machine, but is a mere appendage of successful enterprise, arising from monopolistic or differential advantages inherent in the ingredients of production. This goodwill can be made a maximum only by allocating it to the various plant departments in the manner outlined. Adequate elaboration must be omitted for the present, but it is nevertheless apparent enough that, if this goodwill were set up as a separate entity by determining the economic lives on the basis of the departmental Taylorean unit costs, its value would drop below the maximum attainable by the correct application of the theory of maxima and minima. Two simple numerical examples for a one-department enterprise were given in Part II. Incidentally, I do not determine the market price by external conditions alone, but assume that it is in some manner inversely related to the number of co-operating machines.

6. The whole theory of value and of valuation indeed needs revision. In particular, the role of marginal costs needs increased attention; thus instead of writing for the rent of the machine $R(t)=wQ(t)-E(t)$, we might well write $R(q, t)=wQ(q, t)-E(q, t)$, where q is the number of units of output at time t, and then observe that q is a function at the disposal of the owner in his attempt to maximize his aggregate profit. This is a decidedly more general approach than by variation of T alone.

Answer: This fruitful lead occurs already in Professor Hotelling's 1925 paper and therefore I should certainly have referred to it. As he there states, this approach is applicable where his tentative postulate of use at full capacity is not even approximately true. A further implication is that a reduction of the equipment maintained is not feasible. The idea has monopolistic aspects, which I did point out very briefly in footnote 17 of the paper cited in note 1 above. For this purpose, I have considered q as the unknown ratio of the optimum rate of output to the total capacity.

7. On page 28 you state that "the theory of public-utility regula-

tion is thus quite definite and leaves no room for equivocation." This is so drastic a statement that it is quite likely to be challenged. My own feeling is that, if the statement is true, it is true only as a part of, or as an addition to current legal theory, but it seems to me that the entire basis of this current theory is extremely shaky and will ultimately have to be discarded in favor of the operation of utility plants in the genuine interest of maximum public service. As I pointed out in ECONOMETRICA, for July, 1938, this means a fundamental and drastic change from current theory and practice.

Answer: I agree that my statement should be interpreted only in the light of current legal theory. In fact, I realized that a qualification was in order before this comment reached me and have therefore added footnote 22, which covers the point.

8. You refer to your theory as three-dimensional, with a two-dimensional cross-section corresponding to my work. This is approximately correct but, from another point of view, the number of dimensions in dealing with an infinite chain of replacements might be regarded as infinite. In problems of the continuous variation of the rate of operations also, we have something like a function of an infinite number of variables to maximize. My work on "The Economics of Exhaustible Resources" (*Journal of Political Economy*, April, 1931) is but a special case of this. The appropriate mathematical tool for such cases is the calculus of variations. The case of an infinite chain of replacements has a special mathematical interest because it gives a problem of maximizing a function of an infinite number of discrete variables and thus, in a sense, stands between the calculus of variations and ordinary differential calculus.

Answer: I agree, but wish to point out that comments 5 and 6 above add two further dimensions, so that we now have five in all, in my sense of the word. As additional dimensions are introduced, each previous presentation of the problem of economic life becomes a special case of the last one.

9. It appears that we are now in agreement regarding most points. Our differences seem to be largely concerned with different cases, all of which have economic importance within their respective spheres. This whole discussion illustrates the varied possibilities of setting up a mathematical model to represent the infinite complexities of empirical reality. Different theories arise from emphasis on one or another of the many relevant considerations.

Answer: It seems to me that our cases differ only with respect to the number of dimensions considered in the analysis and in our respective definitions of "static economic conditions." When these differences are reconciled, all apparent contradictions disappear.

10. I suppose that the only thing that can be said about replacement that will be completely general is that an entrepreneur may be expected to try to maximize the present value of all his future net profits.[23] This general statement leaves the way open for innumerable variations regarding assumptions appropriate to special cases. The important thing in building up a theory will thus be to make perfectly clear the assumptions and definitions in each case.

Answer: I agree.

[23] Upon re-reading this appendix, I fear that the term "net profits," as here repeatedly employed, may be misunderstood by those, who are well aware that capital value can not be computed from what accountants call "net profit." Professor Hotelling uses the term in the sense of the Fisherian net income stream, i.e., my formulae (31). This stream includes scrap sales and purchases of new machines, although both items are considered capital transactions in accountancy. On the other hand, it ignores depreciation, although accountants must ordinarily make some such deduction, before arriving at their "net profit." The two concepts are equivalent when the "retirement method" is employed or, regardless of the method of depreciation, in the special case of a mature plant operated under static economic conditions [see formula (30) for $x=0$ and references at end of note 21 above]. Detailed definitions of the numerous "income" concepts in economics and accountancy were given in Chapter II of *op. cit.* in note 16 above. A short review and graphic clarification of the persistent verbal confusion will be found in my recent article on "Economic Theories of Goodwill," *Journal of Accountancy*, September, 1939, p. 175 *et seq.*

THE POSSIBILITIES AND LIMITATIONS OF OBJECTIVE SAMPLING IN STRENGTHENING AGRICULTURAL STATISTICS*

By CHARLES F. SARLE

THERE IS NOTHING NEW or startling in the fact that much better agricultural statistics are needed than can be secured with the facilities and techniques now available. The type of agricultural statistics demanded, at any one time has reflected the nature of the current agricultural problems in much the same way as the economic theories in the past have reflected the current business and economic conditions of the times.

During the several decades prior to the 1860's representative farmers of the nation began to demand timely agricultural statistics showing the "condition" of the growing crops in various parts of the country, production of crops at harvest, and numbers of livestock on farms.

They wanted these agricultural statistics and they wanted them to be issued by a neutral agency such as the Government rather than by trade agencies. Farmers believed that crop reports as then issued by trade interests were juggled for the benefit of the buyers and speculators. They believed that reports of a big crop were put out before a crop was sold by the farmers, only to be followed by reports of a much smaller production after the crop had left the farmers' hands.

The state and county agricultural societies that flourished during this period before the Civil War attempted (without success) to make and issue current crop reports. Co-ordination was difficult and it was impossible to get a sufficient number of farmers to report regularly and consistently. One of the major tasks in 1862 of the newly established government agency that later became the Department of Agriculture was the issuance of official crop reports.

During the half century ending with the World War this nation was a heavy exporter of farm products—cotton, wheat, cattle, and hogs. The demand for agricultural statistics appears to have been reasonably well satisfied with the annual estimates of crop production and inventory numbers of livestock on farms January 1. The need for forecasts of crop production during the growing season was met by the monthly reports of the "condition" or appearance of the crop. These rather crude statistics presumably reflected the direction of year-to-year change in agricultural production and gave a rough approximation of the extent of the change. The trade early developed a simple method for translating the condition reports into forecasts of crop yield and

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production—a method that was later (1915) adopted by the Department.

Prices received by farmers for crops as of December 1 each year were estimated and used in calculating the value of crop production, for each crop and for the total of all crops. Likewise the prices of livestock as of January 1 were estimated and used in calculating livestock values. Furthermore, an inquiry concerning wage rates paid farm laborers—monthly rates, with and without board, and day rates, with and without board—was made 19 times during the period from 1866 to 1909 inclusive, and annually thereafter.

It was not until about 1910 that Nat C. Murray, a foresighted statistician in the Department of Agriculture, succeeded in starting a monthly inquiry on prices received by farmers for 42 agricultural products sold from the farm. In fact, one of the reasons for starting this price-reporting service was to provide useful employment for the clerks during the slack time between monthly crop reports. Annual inquiries relating to prices *paid* by farmers, land values, rents, etc., were begun about this time. One is inclined to wonder just what postwar economists, statisticians, and legislators would have done without these prewar data in calculating price indexes and "parity prices."

During that first half century of official agricultural statistics the methods of sampling employed were relatively simple. Voluntary correspondents were asked to report on the situation in their immediate locality. The sample data represented their judgment as to a given phenomenon—yield per acre for a crop at harvest, condition of a growing crop in per cent of "normal," the current year's acreage of a crop or numbers of a given class of livestock as a percentage of last year's, prices received or paid, prevailing wage rates, and so on. The best and most public-spirited farmers were encouraged to serve as voluntary unpaid reporters.

Accumulative bias in the reports on acreage and livestock numbers became apparent when current estimates in census years were compared with census data. The year-to-year estimates of change applied to the previous census as a base failed to keep abreast of the rapidly expanding agricultural plant during the decades from 1870 to 1900.

Improvements in methods of sampling and estimating during this pre-World-War period were made primarily by: (1) increasing the size of the sample, (2) increasing the number of state agents and extending their employment from part-time to full-time, (3) and employing a few crop specialists on a regional basis. The township list of reporters was set up in 1897, thereby greatly increasing the size of the samples over the size of samples obtained from county reporters. That is, the objective was to have an active reporter in each agricul-

tural township reporting for his locality in addition to a reporter in each county reporting for the entire county. The original plan called for several subreporters to report to the county reporter on conditions in the parts of the county where they lived—a plan that never seemed to work out very well in practice. Furthermore, it was impossible to persuade more than 50 to possibly 75 per cent of the county reporters to respond in any one month. There was obviously a limit to the amount of service these people would give without financial compensation.

The part-time state agents frequently were men on the agricultural staffs of agricultural colleges and experiment stations whose regular duties kept them in touch with agricultural conditions in their respective states. Others were men with political connections. The regional men usually were full-time reporters who traveled over several states. Both state agents and regional men had small lists of crop correspondents reporting directly to them. It was not until 1913 that state crop and livestock estimators were brought under Civil Service.

The entire system of sampling and estimating was highly *subjective* in nature, in that the observations were *opinions* expressed in quantitative terms and the sample was obtained from voluntary crop correspondents who were willing to serve without pay.

The World War with its problems of producing food for civilian and army consumption, as well as for export to the Allies, gave rise to the demand for more accurate estimates of agricultural production and for quantitative forecasts of crop production from the monthly condition reports. In April, 1918 an inquiry on the supply of and demand for farm labor in percentage of normal was started.

The economic and social maladjustments in American agriculture growing out of the World War further increased the demand for more and better statistics, especially those relating to economic aspects of agriculture. Estimates of the annual production of livestock and livestock products and estimates of farm income were started in the 1920's. Estimates of farm employment were begun from samples of the individual farms of crop reporters. The agricultural statistics of the Department began to be used in many ways never anticipated by those who compiled them and who were responsible for their accuracy. Statistical economic analyses were made to determine the factors that influence prices, supply, consumption, and exports of agricultural products. It was during the 1920's that "Big Business" began to demand agricultural statistics of production and income by counties as a basis for determining sales programs, and quotas.

Acute distress arising from the economic effects of the world-wide depression of the 1930's has given rise to many "action" programs by

Federal and state agencies, designed to alleviate directly distress conditions of agriculture. The administration of these programs has called for unexpected uses of available agricultural statistics.

County quotas for cotton and wheat acreages had to be established almost overnight and, to determine them, average yields per acre for the country had to be ascertained. A little later, county quotas for corn and hogs had to be established. Crop and livestock statisticians were drafted for service in connection with the operation of these programs. Other programs, authorized by Congress, called for more and better statistics concerning farm tenure, taxation, credit, and movement of the farm population.

Within the last year the estimates of employment on farms have been used—along with estimates of nonfarm and city employment—for making estimates of monthly employment by states. These estimates of employment have been used for making estimates of unemployment by subtracting the number of employed from estimates of the total labor force. These estimates of unemployment are one of the factors used in ascertaining the work-relief needs among states. Billions of dollars are allocated among states largely on the basis of these crude estimates of employment and unemployment.

With all this increase in the uses made of statistics during recent years little or no provision has been made for improving the existing methods and facilities available for making agricultural statistics. What can be done to strengthen agricultural statistics to meet these ever-increasing demands? As in the past, the answer lies in the further development and use of *objective*, as contrasted with *subjective*, methods of sampling and estimating.

The terms objective and subjective apply (1) to the nature of the individual observations, (2) to the method of sampling followed in taking the observations, and (3) to the methods of estimation used.

Three kinds of observations are made in obtaining sample data relating to agriculture: (1) the *judgment type* of data—the reporter is asked to estimate for *his locality* such items as the condition or yield per acre or change in acreage from last year for a particular crop, or the change in number of different kinds of livestock, or the prices being paid or received by farmers, (2) the *individual farm type* of data—the reporter is asked to report, for the *farm* he operates, the acreage in the various crops, production, or numbers of livestock, and (3) the *direct count* or *measurement* data—where the phenomenon is sampled by either the crop reporter or the state crop estimator himself. The measurement of the acreage or field frontages of the various crops along selected highways by the crop-meter, stand and boll counts for cotton, and head sampling of wheat, are all examples of direct observational data.

The method of sampling followed may be subjective in that the selection of persons reporting or farms obtained in the sample is controlled largely by psychological factors as is the case of the returns from unpaid voluntary crop correspondents. The method may be objective in that the selection of observations is predetermined by the statistician according to sound principles of sampling design.

The method of estimation may be subjective when the statistician depends largely on his (more or less snap) judgment in giving proper weight to two or more indications in making an estimate or objective when the statistician sets up predetermined weights based on evidence as to past behavior, statistical precision of the sample indications, or other well-reasoned criteria for combining two or more indications in making an estimate.

During the decade from 1910 to 1920 under the stimulus of demands for more accurate statistics arising from the World War, the idea of asking the crop reporter for information concerning his own farm was tried out in connection with acreages in the various crops and numbers of livestock on farms—the individual farm inquiry as contrasted with the judgment inquiry applying to the locality. By the middle 1920's the judgment sample on year-to-year change in acreages and numbers of livestock was replaced entirely by the individual farm sample.

At first the reporter was asked to state the acreage of the different crops, or numbers of different kinds of livestock on his farm for the current year and the year previous. It was soon found that this current-to-historic comparison was subject to both *memory bias*—a forgetting of the acreages in some of the minor crops, and to *cash-crop bias*—a tendency so to state the acreages for the 2 years as to minimize the current year's acreage in relation to the acreage in the previous year. The annual June acreage survey, now used in estimating planted acreages, is an inquiry of this kind.

In 1924 the rural-carrier acreage and livestock surveys were started. The rural mail carriers distribute schedules to farmers along their routes. The size of sample was greatly increased. Only data for the current year were obtained. The ratios of the acreages of the various crops to all crop land or to all farm land were used from year to year, or from the current year to the census or base year, in making annual estimates of crop acreages. In this way the influence of any constant bias found in the current-to-historic comparison was avoided.

In the case of the livestock surveys, the year-to-year relationship of the average number per farm was used in estimating changes in numbers of livestock.

By 1929, the method of matching reports from the same farms for 2 successive years was adopted to get an additional indication of year-

to-year change for both crop acreages and numbers of livestock. Stratification of the sample by size of farms and weighting by census distribution of farms by size also were tried out, but the results did not appear to justify the increased work required.

It took approximately 60 years of experience in crop reporting to prove the very limited utility of the judgment sample of year-to-year change in acreage and livestock numbers and to adopt methods of using the individual farm sample. The judgment type of inquiry, however, has proven satisfactory as a basis for estimating prices and wage rates and fairly satisfactory in estimating yield per acre of crops at harvest and condition of the growing crops. The more intelligent, well-informed, and observant the voluntary reporter, the more reliable are the observations in the case of a judgment type of sample. This selection of better-than-average farmers for reporting on judgment inquiries is an advantage.

When these same reporters, however, are asked to report crop acreages and livestock numbers for the farms they operate, this characteristic of selectivity results in a *biased sample.* It is selective, or representative, only of those classes of farmers who will voluntarily reply to a questionnaire they receive in the mail or a schedule that is left in their mail box by the rural carrier. A large number of farmers—nonresident or "suit-case" farmers in parts of the wheat belt, tenants that move frequently, farmers with a poor education or who are not familiar with the English language, and those of a highly suspicious nature—seldom become regular crop reporters and are not likely to respond to the rural-carrier surveys. Farms operated by such farmers are seldom represented in the individual-farm samples relating to acreages and livestock numbers.

It is true that drought, floods, and powerful economic forces are likely to have somewhat similar effects on all classes of farms and farmers but the fact remains that the established methods of sampling that are available to the Department of Agriculture are subjective rather than objective; consequently the samples obtained are selective and not representative of all kinds and classes of farms.

What has been done and what can be done in the way of developing objective methods of sampling?

In the days before good roads formed a network over the agricultural areas and before automobiles became cheap enough for state agricultural statisticians to own, travel was limited to the railroads. A resourceful statistician conceived the idea of getting a measure of the frontage of the fields in the different crops by counting and recording the number of telegraph poles for each crop along the railroad right of way as he rode along on the train. By going over the same route

for 2 successive years he obtained data that would furnish an indication of change in acreage of the various crops found along the route. These were objective observations and although the routes might not be representative, he was able to obtain a sample of all farms along the route—provided of course he could identify the crops in the field as he sped along.

Later the automobile was substituted for the train and telephone poles along the highways were counted or the number of fields in the various crops were counted. Later, about 1925, one of the field statisticians, D. A. McCandliss, invented the "crop-meter" for measuring the frontages in the various crops on one side of the highway; later a "double-bank" machine was perfected that permitted both sides of the road to be "metered" at the same time. Two men in a car are required to measure both sides of the road simultaneously—one to punch the buttons on the crop-meter and the other to drive the car and to check the identification of the crops made by the first man.

The measurement of the frontage of a given crop along a "representative" route taken in the current year is compared with the measurements over the same route for the same crop taken in the previous year or in the census year or in both years, in obtaining an indication of change in acreage. The observational data are objective but the selection of the representative route involves the use of judgment and consequently is subjective and is liable to bias the sample.

A study is being made to test the basic assumptions involved in using the crop-meter data as a basis for estimating acreage change. Aerial photographic survey data for acreage in 19 counties located in 10 North Central States are being used for the purpose of this and other sampling studies relating to acreage. Data for only 1 year have been available for study thus far.

An intensive application of the crop-meter method of sampling is being made in a number of cotton counties where complete AAA acreage measurements will be available for checking the sample.

The crop-meter sample has proved to give a valuable indication of year-to-year change in the acreage of a crop that changes greatly from year to year in an important commercial producing area, such as potatoes on the Eastern Shore of Virginia. The crop-meter sample has given a reliable indication of the acreage of cotton in South Carolina and has been used for a number of years for cotton in other states. But in some areas, such as the lowlands of river valleys, the cropping practices and selection of crops along the highways might not be at all representative of those away from the roads.

The crop-meter cannot be used in estimating changes in livestock numbers. The samples of individual farms obtained from voluntary

crop correspondents and by the help of the rural mail carriers are not sufficiently representative to meet present-day demands for accuracy and for geographic detail involved in county estimates.

The logical way out of this difficulty in getting selectivity of acreage and livestock samples is to provide for an *enumerative sample* or sample census each year. Paid enumerators would obtain acreages of the various crops and the number of the different kinds and classes of livestock from a sample of individual farms by personal interview. Dependable statistical methods could be used in designing a sample that would be representative of all kinds of farms in an area. This method of conducting an enumerative sample has been tried experimentally with good results, but it is much more expensive than obtaining a sample by mail or with the help of the rural carrier.

When samples are used instead of a complete enumeration, there is always the problem of having a sample large enough to give "serviceable accuracy" to the average or other statistics calculated from the sample, assuming, of course, that the sample is not selective and is free from bias. Data relating to the acreage of a given crop or numbers of a particular class of livestock on individual farms vary greatly from farm to farm even in the same locality and from year to year on the same farm. For statistical purposes the acreage of a crop on some farms may be "zero," on others it may be hundreds or even thousands of acres. Some farms may not grow the crop at all one year and may have a substantial acreage of that crop in another year. Variability in acreage of a crop has no well-defined limits established by nature as is the case with yield per acre of a crop.

The problem of making reliable estimates of yield per acre for various crops does not involve so many serious difficulties as the problem of estimating acreage. The well-informed crop correspondent is usually well qualified to estimate the yield per acre for a given crop on his own farm and usually on a number of farms in the neighborhood. It is entirely practicable, therefore, to use these "judgment" (for the locality) reports on yield as the direct basis for annual estimates of yield per acre. However, the figures for yields per acre derived from census data serve as a useful check in census years and as a measure of bias that is applied to the current year's sample. Any incompleteness in the census coverage of farms is not likely to be serious, in so far as yield per acre is concerned, unless the enumerator consistently fails to get either the lower- or the higher-yielding farms in his district. But the nearer the time of taking the census to the completion of harvest, the more reliable are the census data on yield per acre. Late fall would be the ideal time to take the Federal census of agriculture—before farmers have moved to other farms.

The problem of selectivity of the sample is much less serious with the judgment inquiry on yields than with the individual-farm inquiry on acreage. Selectivity, however, must be guarded against. In recent years the regular judgment inquiry on yields has been supplemented by an individual-farm inquiry on acreage and production, from which a figure for yield per acre is derived. As might be expected, such an inquiry is selective of the more productive farms in most states and the yields obtained are used primarily in a relative way to indicate year-to-year changes. Judgment yield inquiries obtained from operators of grain elevators also are used in some states to supplement the returns from regular crop reporters.

The yield-per-acre data, whether from the judgment inquiry for the locality or from the individual-farm inquiry, are much less variable than are acreage data. Nature sets up limits to the variation in yield per acre. Consequently, the size of sample need not be nearly so large for a given degree of precision as with samples of crop acreages or livestock numbers on individual farms.

The greatest weakness or source of error in the data and methods now in use in estimating yield per acre is the cash-crop bias (understatement) that appears in the sample data for certain, if not all, commercial crops. This cash-crop bias is not consistent from year to year in the same state or among states in the same year.

To meet the problem of cash-crop bias in yield sample data the Department of Agriculture is experimenting with preharvest field surveys for cotton, wheat, corn, and apples. There is a need for further development and use of such methods based on physical measurements of yield taken from representative fields whereby the factor of human bias is held to a minimum or is eliminated. Additional trained crop estimators are needed to obtain a sample of sufficient size, as these surveys involve actual counts and measurements—size of ear in the case of corn, head samples for wheat, boll counts and size of bolls for cotton—made in commercial fields over wide areas.

In the summer of 1938 a survey of wheat was made immediately preceding the harvest in five crop-reporting districts in Eastern North Dakota. East-and-west and north-and-south routes were selected in each district. Wheat fields were selected every 2 miles of wheat frontage on one side of the road along these routes. A crop-meter was used to measure the wheat frontage. Two head samples were taken from each field from a strip not over 125 paces wide that paralleled the highway. In all 232 fields were sampled. Seven varieties were found, although 444 of the total of 464 samples were of three varieties—Ceres, Thatcher, and Durum.

The head samples were threshed later and the yield data were

analyzed to determine sources of variability. The greatest source of variability was *variety*, the next largest source was the geographic districts. The variability among fields was significantly greater than the variability within fields, indicating that maximum efficiency in sampling could be obtained by increasing the number of fields rather than by increasing the number of observations within fields.

The yield obtained from the objectively taken sample was generally higher than the yield reported by agricultural correspondents. This difference is no doubt due in part to losses in grain that occur in harvesting and threshing or combining. A similar preharvest survey is now (1939) being taken extending from Oklahoma to the Canadian border. Quality characteristics of protein percentage and test weight per bushel also are being ascertained.

Estimates of production are made by multiplying the estimates of harvested acreage by estimates of yield per acre. In the case of a number of commercial crops, marketing or processing records are obtained and used in truing up estimates of production for the previous year. A complete record of the quantity of cotton ginned each year is made by the Census Bureau. Reasonably complete information is obtained on total annual sales of such crops as tobacco, peanuts, rice, sugar, and crops for canning. Railroad shipments and mill-door receipts are obtained for wheat, rye, and flax in a number of States.

When the total quantity of a crop sold is considered along with estimates of farm utilization for seed, carry-over on the farm into the next season, and other farm uses, an estimate of production can be made for a crop that is independent of the production estimate based on acreage and yield per acre. These estimates based on sales are useful in assisting the crop estimator in finding and measuring bias in the sample data and methods used in making the current estimates. Every effort is made to obtain data on sales, market movement, and processing of commercial crops.

FORECASTING CROP PRODUCTION

The first official forecasts of crop production for most of the major crops are made in connection with the July crop report. The production of fall-sown winter wheat and rye is forecast several months earlier, and cotton on August 1 each year.

The forecasts of crop production, exclusive of tree and bush crops, are made in two or three steps: (1) estimates of acres planted, (2) current estimates of abandonment of acreage since planting for certain crops, and (3) forecasts of yield per acre.

Forecasts of yield per acre before harvest are based primarily on reports of the condition, or promise of yield, of the crop "in per cent of

normal" obtained on the first of each month from the regular crop correspondents of the Department. Later in the growing season a report on the probable yield per acre also is obtained. Both of these samples are judgment inquiries based on the mass opinion of the reporters and they apply to the locality in which the crop reporter lives.

The condition reports are not so likely to be selective as are the reports on yield per acre or acreage. The condition data also are less variable. The very nature of the condition inquiry leads to less variability in the individual observations. Normal yields may vary considerably from one area to another, even within a county. A condition of 50 per cent would mean a yield of 20 bushels of corn in one area and perhaps 25 or 30 bushels in another. In fact, the greatest variability ordinarily shown by a condition sample is about equal to the smallest variability found with judgment yield sample data (the reports on yield). The most variable yield samples seldom exceed the variability of the least variable samples of individual farms on acreage or livestock numbers.

A forecast of yield per acre is made on the basis of the statistical relationship between the reported condition for a given date and the final yield per acre shown for a period of 15 or more past years. The relationship between November 1 condition and final yield per acre for cotton for the period from 1922 to 1936 for the United States is shown in Figure 1. The average relationship in this case is best measured by a straight line, called a regression line. If the currently reported condition is, say, 60 per cent, then a forecast of a yield of 172 pounds per acre is indicated by the regression line. The years 1937 and 1938 do not fit in with the relationship shown for the previous years. This interesting point will be considered a little later.

One of the fundamental difficulties in using the reported condition of the crop as a basis of forecasting yield lies in the subjective nature of the sample data—the mass judgment of a large number of crop reporters. There are times when one or more serious insect pests or plant diseases are present in a crop but are not sufficiently in evidence to be readily recognized by anyone not well trained in entomology and plant pathology. Stem rust in wheat is a good example of a disease that is not readily recognized in its early stages of infestation. Under such condition, the crop reporters' estimates of condition are likely to be entirely too high in terms of the final yield of the crop.

One way to meet this difficulty is to have crop estimators, trained in the identification of insect pests and plant diseases and experienced in estimating damage to yields resulting therefrom, make frequent field inspections of the growing crop in important commercial producing areas.

The crop reporter's concept of a normal or full crop—condition is reported in per cent of normal—remains reasonably constant over a long period of years. In fact, the crop correspondents are slow in changing their concept of normal when the introduction of improved cultural practices or higher-yielding varieties call for an upward revision. For example, the increasing use of soil-building crops and the concentrating of reduced cotton acreage on the more fertile land on the

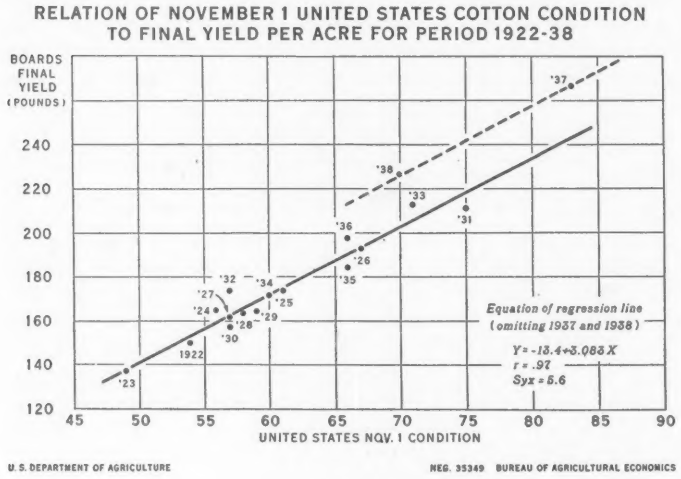

FIGURE 1

farms of the South have resulted in substantially higher yields of cotton. The reported condition or appearance of the growing crop apparently tends to reflect the vegetative more than the reproductive aspects of the cotton plant. The reported condition for cotton has failed to allow for the increase in potential yield—the level of the line of relationship between November 1 condition and yield for 1937 and 1938 has apparently been raised some 20 pounds. This apparent change in the level of relationship in the last 2 years is shown graphically in Figure 1. This difficulty is partly overcome by asking the crop correspondent to report in terms of probable yield per acre.

The rapid introduction of hybrid corn in the Corn Belt States has had much the same effect in raising the line of relationship for corn condition and yield. The rapid expansion of soybean production in states like Illinois during recent years, accompanied by the selection

of the higher-yielding varieties, has resulted in a sharp upward trend in the condition-yield relationship for that crop.

Not only is it possible for the level of the condition-yield relationship to change but the relationship can even change from one with a *positive* slope (the higher the condition the higher the yield) to one with a *negative* slope (the lower the condition the higher the yield). Such a change occurred in the case of winter wheat in Maryland soon after 1900. The disease known as septoria nodorum became widespread in Maryland at about this time. It is not readily recognized during the growing season and does its greatest damage under weather conditions that are highly favorable to the vegetative growth of the wheat plant. Consequently a high condition is then associated with a low yield.

Potatoes and peanuts are crops that do not show a high degree of relationship between the reported condition of the growing crop and the final yield. This is not surprising in view of the fact that potatoes and peanuts are both developed underground and the reported condition of the crop tends to relate largely to the appearance of tops or vines. Obviously it is essential that more precise methods be developed for forecasting yield per acre in advance of harvest for these two crops. Some research has been undertaken (financed by the special research funds allotted by the Secretary of Agriculture for this work) to develop methods of forecasting crop yields of a more objective nature. The experience gained in making boll counts of cotton and the results obtained in studies of the relationship of yields to weather indicate three lines of development: (1) methods involving the use of weather data along with condition in forecasting yield, (2) methods based on the relation of yield to structural counts and measurements of the plant characteristics associated with yield from representative samples of the growing crops, and (3) methods involving the use of the direct or indirect influence of weather on crop yields.

The reported condition as of the first of a particular month may not fully reflect the full influence of weather that has prevailed up to that time. A deficiency in subsoil moisture, accompanied by adequate surface-soil moisture, may give a temporarily high condition for a crop at a given date. If rains are ample and well distributed, a good yield may result, but normal or light rains may not furnish sufficient moisture to carry the crop through to a good yield.

The yield of a crop depends on certain plant characteristics such as stand, size of the plant, size of the head or ear, and plumpness of the kernels. Some of these characteristics can be measured well in advance of harvest. The crop-weather research studies regarding wheat, carried on in Great Britain, indicate that height of plant just prior to

heading is positively correlated, and maximum number of tillers early in the season is negatively correlated, with the yield of wheat.

The analysis of head samples of spring wheat representing the eastern half of North Dakota, taken immediately preceding the 1938 harvest mentioned above, shows that height of the wheat plant, number of heads, and length of heads, are all associated with yield per acre. In the case of Ceres multiple correlation was $R=0.84$, with Durum $R=0.86$ and for Thatcher $R=0.95$. These are plant characteristics that could be measured several weeks before harvest, and at a time when the prevalence of insect or plant disease could be detected by a well-trained observer.

For about 10 years, during August and September, the Department of Agriculture has been making boll counts and measurements on cotton in commercial fields along several thousand miles of routes laid out through the Cotton Belt. For the last 2 or 3 years, field observations have been made on corn in the major Corn Belt States during September. The results obtained suggest the feasibility of developing objective methods of *forecasting* the yield of these two crops based on actual field observation.

In a number of states, studies have been made of the relationship of weather factors to the yield per acre of various crops using averages of weather data and yields for the entire state. Although some of these studies have shown a high relationship for the period of years covered, the forecasting formulas developed have not proved very satisfactory when actually used for forecasting in seasons not included in the study.

During the last 2 years, research of a more intensive nature has been undertaken using weather and yield data from the agricultural experiment stations. Moreover, special field-plot experiments designed for the purpose of studying the effect of weather and soil moisture on growth and yield of wheat, corn, and cotton have been started—one on winter wheat at Manhattan, Kansas, one on corn at Ames, Iowa, and another on cotton at Florence, South Carolina. Methods are being developed for doing this type of crop-weather research

Five or six years of crop-weather field experiments at some 6 to 10 stations for a given crop would form an excellent basis for ascertaining the direct and indirect influence of weather on crop growth and yield per acre and perhaps would make it possible to develop objective methods of forecasting yield per acre from weather.

One of the inherent difficulties in forecasting the yield per acre in advance of harvest is that the growing and maturing crop is always subject to the hazard of damage either directly from weather or indirectly from disease and insects which are, in turn, largely controlled by weather. On the other hand, the yield of a crop with indeterminate

fruiting habits, like cotton, can be greatly increased even late in the season by unusually favorable weather.

The earlier in the growing season the forecast is made the greater is the hazard of subsequent weather. It is not surprising, therefore, that early season condition usually is not sufficiently related to final yield to justify its use in forecasting. For many years, the condition of cotton was obtained as of May 25. In Figure 2 it will be seen that the rela-

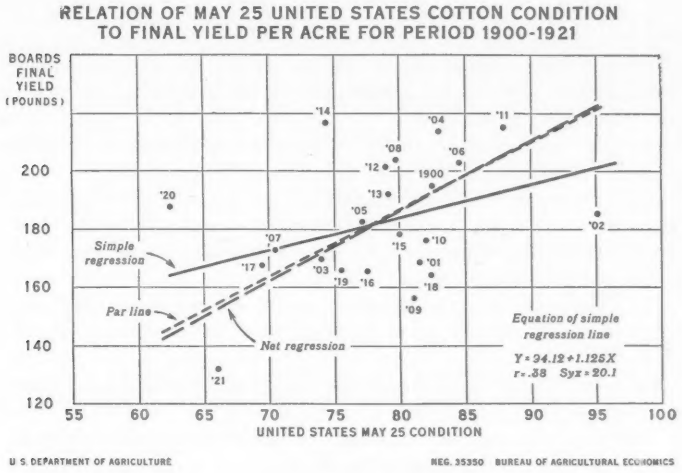

FIGURE 2

tionship of condition with final yield was too low ($r=0.38$) to be statistically significant and that condition at that time would be valueless in forecasting yield. The dotted line on this chart is the regression line when the change in condition from May 25 to October is taken into consideration. The dashed line is the "par" line formerly used in interpreting condition. No condition data are now obtained nor are forecasts made of the cotton yield per acre before August 1.

If accurate forecasts of yield per acre are to be made early in the growing season, it will be necessary: (1) to know the relationship between weather and yield per acre, and (2) to know how to forecast the weather, or at least the extremes of weather, for several months in advance.

A research program in meteorology that has practically doubled the scanty knowledge in this field has been under way during the last

2 years. It will take time and a great deal of investigation to develop dependable methods of long-range weather forecasting.

CONCLUDING OBSERVATIONS

The ever-increasing demand for more and better statistics relating to agriculture in its many phases can be met by the two governmental agencies responsible for these statistics (the Agricultural Marketing Service of the U. S. Department of Agriculture and the Bureau of the Census of the Department of Commerce) by the development and practical use of objective methods of sampling and estimation.

Three specific suggestions for using objective sampling methods in strengthening agricultural statistics were made by the writer in a previous paper,[1] as follows:

(1) *Use a split schedule* in Census enumerations to broaden the scope of basic agricultural statistics by from 50 to 200 per cent.

(2) Take a *rotating partial census* in intercensal years in commercial fruit and truck areas to obtain more reliable and more useful basic statistics regarding these specialty crops than is possible in connection with a general enumeration.

(3) Take a nation-wide annual enumerative *sample* of individual farms covering crop acreages and production, livestock numbers and production, etc.

The need for these improved methods has been developed more fully in the present paper, in which a critical description has been given of some of the more important methods of sampling and estimating now used by the U. S. Department of Agriculture and of some of the results obtained in developing more objective sampling methods.

In looking into the future it is not difficult to visualize a return to the county statistical reporter as the source of much of the sample data on which estimates concerning agriculture would be based. He would not be asked to render his service free of charge. He would be given part-time employment and would lose his job if he failed to make the required reports regularly and well. Very little of the data he would submit would be based on his judgment or opinion alone.

Once or twice a year he would take an enumerative sample of individual farms in his county—an objectively designed sample of representative farms. He would obtain information concerning these farms as operating units—crop acreages, production, and sales; numbers, production, and sales of livestock. Each month he would have variously assigned duties—the taking of head samples of wheat in one month,

[1] "Development of Partial and Sample Census Methods," *Proceedings of the Farm Economic Association Meeting*, December, 1938.

cotton-boll counts or ear counts for corn in another month, and perhaps depth of soil moisture another month. Each month he would report on prices received and paid by farmers, numbers of persons employed, and perhaps rate of egg and milk production, as well, on sample farms within the county.

He also might obtain a record of purchases of farm products by local buyers and processors and shipments out of the county by rail and truck of agricultural products to be used with other data in estimating farm income by counties.

Such a program would be more expensive than the present system of voluntary unpaid correspondents, but the resulting statistics would be worth many times the increased cost in the more efficient administration of state and Federal action programs alone.

United States Department of Agriculture

RESIDUAL, DIFFERENTIAL, AND ABSOLUTE URBAN GROUND RENTS AND THEIR CYCLICAL FLUCTUATIONS*

By Karl Pribram

I. THE "STATIC" THEORY OF THE URBAN GROUND RENT

The objections which have been raised against the fundamentals of the value theory of the classical school of economics hardly ever went so far as to affect the principle upon which the Ricardian explanation of land values was based, viz., the idea that the value of land is not to be explained in terms of cost, but in terms of a residual rent. This idea when developed into its consequences has even facilitated the insight into many value phenomena other than land values.

However neither Ricardo nor his postclassical followers ever attempted thoroughly to examine the question whether and with what modifications the residual rent concept can be used in interpreting the economic phenomena presented by the existence of *urban land* values.

To my knowledge the first author who undertook the task of establishing a consistent theory of the urban ground rent was the Austrian economist F. von Wieser.[1] Wieser emphasized an important distinction obtaining between the agricultural ground rent and its urban counterpart with respect to the relations between the factors which are instrumental in producing the differential returns: the former is due to the fact that, owing to increased demand for farm products, agricultural production may be profitable also at increased costs, with the result that a surplus profit accrues to those farmers whose production is privileged by lower costs. On the other hand, according to the Wieserian theory, the residual rent of urban sites is largely independent of differences in costs of construction; it is due to differences in the demand for the services rendered by different real-estate properties and, consequently, to differences in the prices paid for such services.

Von Wieser has specified his theory by distinguishing within each locality a range of housing markets which differ according to the situation of the sites. The tendency to exploit the privileged sites to the fullest possible degree is reflected in the process of intensifying the utilization of land of this type both horizontally and vertically. The larger the portion of the lot which is utilized for dwelling and renting purposes and, in addition, the larger the number of dwellings and premises

* Paper presented at Colorado Springs, July 26 and 27, 1939, before Cowles Commission Fifth Annual Research Conference on Economics and Statistics, under title of "Some Dynamic Aspects of the Urban Ground Rent."

[1] v. Wieser, "Theorie der gesellschaftlichen Wirtschaft" in *Grundriss der Sozialökonomie*, I, p. 362.

which are superimposed in increased numbers of stories, the more differential returns are adding up and are attributed to the land, thereby increasing its value. Thus the increased rent of farm land which is due to intensified cultivation is paralleled by increases of the urban land rent which result from intensified utilization of urban sites. In both cases this process of intensified utilization is limited by the law of diminishing returns.

The distinction between the concept of residual rent and that of differential rent can be drawn as follows: The *residual* rent is "the amount remaining out of joint land and building income after the income attributed to the building is subtracted. . . . Any surplus return on the land and building above that necessary to pay operating expenses, taxes, interest, and depreciation on the building is properly assigned to the nonreproducible element in the combination—land."[2]

The concept of *differential* rent, however, emphasizes another aspect of the same phenomenon, the difference which exists between the actual or prospective net income resulting from the utilization of an urban site and the net income of the marginal site which yields no urban ground rent. Hence, the differential urban ground rent is somewhat lower than the residual rent; it is equal to the residual urban rent minus the agricultural ground rent which is to be paid, in capitalized form, to the owner of the farm land when land is purchased for urban uses. Thus the value of the farm land constitutes a cost element of the urban real-estate property. In the course of this process the rent of the farm land loses its original residual character.

It is a remarkable feature of this theory that it is derived, on principle, from a primarily static concept of economic system or rather from a concept based on the idea of continuously changing equilibrium positions. In this respect it is exactly the counterpart of its Ricardian prototype. When pursued into its logical consequences the theory leads to a conclusion similar to that drawn by Ricardo with respect to the rent of agricultural land: that there should exist at any time a certain category of suburban sites which have not reached the stage where the development of the ground rent begins. Sites of this type should particularly be found at the outskirts of the cities where plenty of agricultural land is available in quantities equal to the quantities of the reproducible elements of residential production.

There is no doubt that the application of the Ricardian rent concept to urban sites is helpful in explaining rent differentials and differences in land values which result from the existence of more or less strictly separated housing markets and from the intensified utilization of

[2] Homer Hoyt, *One Hundred Years of Land Values in Chicago*, Chicago, 1933, p. 451.

certain privileged sites. However, it is doubtful whether the existence of marginal urban sites whose value is exclusively determined by the level of the agricultural rent is borne out by the experience and practice of the real-estate market. In many European cities the attempt to explain urban land values exclusively in terms of differential rents was challenged by the erection of large tenement houses at the outskirts of urban agglomerations. If no other factor but privileged location were instrumental in bringing about urban ground rents, this phenomenon would be not less striking than intensified utilization of poor and remote farm land in a country with a large supply of uncultivated arable land. If the rentals of every tenement-house dwelling would yield, in addition to the interest and replacement of the costs of construction, only such a contribution to the ground rent as is strictly proportionate to its share of the former agricultural rent, no economic advantage could be obtained by increasing the number of dwellings through the erection of multiple-story buildings. Such a procedure would yield no profits which could not equally be derived from the construction of one-story buildings. The "intensity" rent is a derivative of the differential rent and increased intensification of construction can be profitable only if a differential rent is secured by the privileged location of the site. It is therefore to be assumed that also the land in the poorest location, at the outskirts of the town, is not marginal land in the sense that it does not yield any urban ground rent; the rentals of the dwellings erected upon it are likely to yield a larger net return than would be needed to cover the agricultural ground rent.

It may be argued that the existence of large tenement houses at the outskirts of European cities and the sudden increases in the value of farm land which ripens into residential uses could be explained by a process of anticipating future increases in rentals and, consequently, future increments of the ground rent. But also this problem can hardly be explained in terms of the Wieserian theory. Unless it is assumed that every suburban area is going to enjoy the advantage of privileged location, this theory provides no foundation for the belief that every urban lot is secured increases in value. Yet this belief is so strongly rooted in public opinion that in this country up to recently in the majority of mortgage deeds no provision was made for amortizing the costs of construction. Failure to provide for such amortization can only be justified if there exists a high degree of probability that progressive depreciation and obsolescence of the structure will be made up by the increasing value of the land upon which the building has been erected. Mortgage practices of this type were strongly objected to in Europe; but in Europe also it is commonly held that urban sites of all categories are headed for increases in value.

In order to explain this belief it is necessary to show that there are economic phenomena which are likely to produce urban ground rents of a third type which may be termed "absolute" ground rent. The concept of absolute ground rent implies that any land within the suburban area, when used for housing purposes, is likely to yield a rent higher than that which had been produced through its agricultural use; moreover that, in the long run, the ground rent of urban land of every type is steadily increasing.

The Ricardian doctrine that there exists an ever-increasing absolute agricultural ground rent which will absorb an ever-increasing portion of the national dividend was predicated upon the Malthusian theory in terms of which scarcity of the farm land is bound to increase. If it cannot be assumed that—apart from land in particularly privileged urban districts—scarcity of urban land is equally increasing, the ever-increasing absolute urban ground rent cannot be accounted for by a similar theory based upon disproportionate increases of the urban population and of the land available for its housing needs.

Under these circumstances it is suggested to adopt a somewhat different approach which takes account of the dynamic features of the economic system and centers on the analysis of the influences which might be exerted on urban land values by successive fluctuations of business activity. There is no doubt that these cyclical fluctuations are accompanied by corresponding more or less pronounced increases and declines in land values. What matters for the problem on hand is the question of how the economic effects of these fluctuations on land values can be interpreted with a view to explaining a general upward trend of land values.

II. THE DYNAMICS OF THE URBAN GROUND RENT

A. EUROPE

This somewhat intricate question may be discussed in the first place in terms of European building conditions which are, in many respects, far simpler and easier to analyze than the more complex conditions obtaining in this country. It is particularly important to emphasize that, as distinct from the cyclical fluctuations of American building activity, the corresponding European fluctuations do not reveal the picture of long cycles extending on an average over 18 years. The European building cycles are clearly determined by the same forces which are instrumental in generating business fluctuations or, to be more specific, were thus determined during such periods in which free development of building activity was not interfered with by rent-restriction legislation, schemes of subsidized construction, and similar de-

vices. Hence it is necessary to resort to prewar figures in order to get a pure picture of typical European building cycles.

The prewar pattern of residential construction is almost exactly the same for Germany and England, two countries for which satisfactory figures are available.[3] This pattern shows that building activity, although clearly determined by the course of general business fluctuations, was marked by movements which anticipated by considerable periods (one to three years) changes in expansion or contraction of business.

Thus in Germany building began to increase rapidly in 1901 when business was depressed and remained high until 1906 when general recovery had developed into prosperity. When this stage of business activity had been reached, house building fell off rapidly; the decline started one year before the peak of the business boom had been attained. In 1909, when business activity was at a low ebb, building recovered and increased up to 1911; then it started declining and sank to a low level in 1913, whereas general productive activity simultaneously had reached a new peak.

A similar picture is presented by the prewar British building cycles. Residential construction began to rise in 1902, that is two years after the peak of the business cycle; the recession in building activity started in 1905 when general prosperity was well under way culminating in 1906 and 1907. In 1909 a sharp decline of building activity set in while business activity was expanding. Building reached its bottom in 1910, when business activity had attained a new high level; falling off of general business activity between 1910 and 1912 was accompanied by a revival of building activity. In 1912 business activity improved and business activity was relatively low.

This behavior of building activity has been characterized as anticyclical, that is to say moving contrary to the directions adopted by the business cycle. It has been held on good grounds that residential construction thus contributed considerably towards counteracting the effects of business fluctuations and preventing them from assuming exaggerated proportions. The reasons for this remarkable phenomenon are likely to be found in the fact that the movements of European building activity were largely determined by the level of the interest rate and costs of construction.[3a] In spite of many differences the European mortgage markets were in so far structurally alike as low interest

[3] See, for short analysis, L. Grebler, "House Building, the Business-Cycle and State Intervention," *International Labour Review*, March, 1936.

[3a] See also the results of correlation calculations made for series of residential building of a number of countries by J. Tinbergen, *Statistical Testing of Business Cycle Theories*, Vol. I, Geneva, 1939, pp. 90 ff.

rates prevailing on other capital markets stimulated investment in building, particularly when, at the depth of a depression, this situation coincided with relatively low levels of costs of construction. The most recent British building cycle appears to have reproduced approximately the prewar pattern. Building activity increased rapidly in 1933 when business was depressed and reached its climax in 1936 when prosperity was well advanced.

What may have been the relations between such cyclical movements of building activity and changes occurring in the urban ground rent? It may be recalled that the differential rent which is due to location and intensity was explained in terms of comparisons based upon coexisting returns earned from buildings which had been erected at the same costs per unit of construction. However, this explanation neglects the fact that a residual rent may also be brought about by differences in the costs per unit of construction if such costs are changing in the course of building activity and that similarly the development of the ground rent will be affected by changes in the gross returns on the buildings which are bound to result from business fluctuations and ensuing fluctuations in the volume of the national dividend and distribution of incomes. Particularly heavy fluctuations of the residual rent may be produced by disproportionate movements of gross returns and costs of construction; since business fluctuations are quite generally characterized by strongly pronounced price dispersions, occurring on an upward scale in prosperity periods and on a downward scale during the downswing, it can offhand be expected that during both phases of the business cycle the urban ground rent will vary under the influence of disproportionate movements of the monetary factors which are instrumental in determining its size.

The disproportionate movements of these monetary factors in the course of a typical European building cycle can be described as follows: Failure of the rentals to adjust themselves to rapidly increasing costs of construction during the later stages of business expansion rendered building activity unprofitable when prices of building materials and wages in the building trades experienced sharp upward movements owing to the increased demand resulting from commercial and industrial construction; at the same time the mortgage market tightened since investment in industrial shares offered better prospects than investment in building and saving societies or in long-term bonds issued for financing residential building. Consequently residential construction perceptibly slowed down during boom periods and reached a low level long before business activity had reached its peak; the number of vacancies fell off and occasionally even outright housing shortages occurred. Such a situation was likely to produce almost

general increases in the rentals of older dwellings since it coincided with general wage increases resulting from the progressive adjustment of wages to rising prices. It thus happened that frequently the costs of construction of recently created dwellings eventually established the standard for the return of the houses erected at lower costs during previous periods of building activity. Since the operating costs of the older buildings did not increase proportionately to rising gross returns, the net yields of the older buildings were likely to rise substantially. It is to be emphasized, however, that the urban ground rent was, on principle, not affected by this process, although rising rentals might have resulted in sending up real-estate prices. But on closer calculation it was apparent that such price increases simply meant adjustment of the value of the existing houses to increased costs of reproduction; this operation did not, in fact, affect the urban ground rent and its derivative, the value of the urban land. Boom periods were largely instrumental in enhancing the demand for privileged sites and thus in producing differential rents; but they were not responsible for all-round increases in the urban ground rent.

The time for the general rise of urban ground rents arrived when a subsequent depression had exhausted its cumulative effects, and building activity started again—as already mentioned a long time ahead of any general expansion of business activity. Since rentals had hardly substantially followed the general downward trend of prices, gross returns of the dwellings were still approximately determined by the level of such costs of construction as had obtained when building activity had fallen off in the course of the preceding prosperity period, whereas it was now possible to build new houses at considerably lower costs. Thus a substantial difference existed between the gross returns of the newly constructed houses and the current annual expenditures to be spent on such houses; this difference which was a feature common to almost the entire housing market assumed the nature of an absolute increment whose existence can be explained in terms of changing costs of construction.

Since almost all urban sites participated in this process of increasing ground rents, regardless of their location, an increment in value, over and above the value due to the use as farmland, resulted for all suburban lots that were situated at the outskirts of the city, even those which, from the point of view of differential rent, presented no particular advantage. Thus the erection of multistoried buildings in any suburban district was likely to be a profitable business.

If the preceding analysis of the European building cycle is, on the whole, correct, it is likely to reveal the central position occupied by the absolute urban ground rent in determining the course of building

activity. Building activity started increasing when and because the absolute ground rent was high; it slackened when construction of new dwellings, unless undertaken on privileged sites, offered little prospect of securing a satisfactory ground rent. Since gross rentals, interest rates, and costs of construction are the main factors whose movements are instrumental in determining the urban ground rent and since these factors are primarily influenced by business flucuations, it is evident that the European building cycles were originated and actuated by the same dynamic forces which generated business cycles, although the course of the building cycles tended to overlap that of the business cycles.

This functional relationship of the two cyclical phenomena appears to have exerted a leveling influence on each of them and although the housing conditions of the large masses of the working population were far from being satisfactory, unless taken care of by subsidized schemes, at least real-estate values were to a considerable degree protected against heavy depreciation, and investment in buildings compared favorably with gilt-edged securities of the highest type. Among the various features of the urban ground rent those connected with the location of the sites and intensified construction have primarily attracted general attention, but the absolute ground rent appears to have been far more important economically, since its movements were responsible for the course of general building activity.

B. UNITED STATES OF AMERICA

Since the dynamic economic forces are much more violent in this country than elsewhere, it is small wonder that almost all features of the urban ground rent have assumed far greater proportions than in Europe. Not only are the fluctuations of the urban ground rent far more pronounced but also rent differentials resulting from location and intensity of land utilization have attained amazing ratios, particularly since methods of construction have permitted the erection of skyscrapers and utilization of preferred sites to a degree unknown before.

The dynamics of American city developments have even led to a particular phenomenon connected with that type of ground rent which results from differentiated residential uses to which urban districts are put: the so-called real-estate cycles of residential districts. It is a well-known fact that many residential districts are built up in relatively short periods of time, in the course of a process which is accompanied by rapidly rising ground rents and land values, but that this process is frequently followed by another resulting in gradually declining values until finally the district becomes a blighted, decadent area

or even a slum. The duration of the complete decline of such residential districts has been estimated to range between 20 and 75 years. The existence of real-estate cycles of this type is likely to discourage the common belief that population growth secures continuous increases in ground rents and land values of all urban sites.[4]

Far more striking differences between European and American conditions are found to exist when the cyclical fluctuations of urban ground rents and land values are made the object of comparative analysis. The view prevails that, as distinct from their European counterparts, cycles in American building activity are not primarily, if at all, produced by the forces which are instrumental in determining the course of general business fluctuations. This view is based upon studies made into the course of building activity in individual localities; such studies have evidenced the existence of long cycles whose duration ranges between 15 and 20 years; corresponding cycles are found in subdivision activity and real-estate trading. Since the local cycles, although differing among themselves as to length and intensity, are fairly well co-ordinated as to their timing, it has been possible to establish clearly defined long cycles of national construction. These cycles also are obviously much longer than the cyclical fluctuations of business activity and appear to develop to a large degree independently of the latter.

Comparisons of an index of building costs with the fluctuations in the volume of building activity does not seem to lend support to the hypothesis that building costs are the controlling factors in constructional activity. Such connection as was found suggests that changes in building activity cause changes in building costs rather than vice versa.[5] In an examination of six building cycles made by Warren and Pearson it was shown that in two cycles (1864 to 1871 and 1878 to 1894) rising building activity was accompanied by declining prices; in four cycles (1843 to 1853, 1898 to 1905, 1908 to 1910, and 1921 to 1925) rising building activity coincided with rising prices. Particularly the last building boom was marked by rapid advances in the cost of labor which increased by approximately 30 per cent between 1921 and 1929.[6] Similarly it is held that major cycles in building activity are not explained by changes in money-market conditions.[7]

With respect to changes in land values it is stated that building

[4] See F. M. Babcock, *Valuation of Real Estate*, New York, 1932, pp. 75 ff.

[5] W. H. Newman, "Building Industry and Building Cycles," *The Journal of Business of the University of Chicago*, Vol. 8, July, 1935, Part 2, p. 24.

[6] G. F. Warren & F. A. Pearson, *World Prices and the Building Industry*, New York, 1937, p. 173.

[7] W. H. Newman, *op. cit.*, p. 31.

booms are not initiated by such changes but that land values tend to follow a major rise of building activity thus becoming a cumulative factor in such a boom; on the other hand, since land prices are slow to move downward, they contribute to the break of the building boom when they are eventually found to have no counterpart in increased earnings and subsequently they tend to increase the severity of the decline of building activity.[8]

The view that a close relationship is lacking between building cycles and business fluctuations has been given support by the observation that the amplitude of the oscillations of building activity far exceeds any similar amplitude in changes of general business activity. The building peak may be 1500 per cent above the trough; and quite generally it has been stated that the cycle of the residential building industry ranks among the largest and most extreme.[9] Hence, in order to explain periodic movements of such magnitude and intensity, factors other than those responsible for business fluctuations have been looked for. Among these factors primary attention has been focussed upon shifts in population and the events associated therewith. An explanation in terms of population movements has been suggested by W. H. Newman;[10] the same idea has been elaborated in an able article by Professor A. Burns. He has analyzed certain basic factors in the fluctuations of residential construction: durability, inconstancy of the housing standard, the fluctuating rate of population change, and immobility. In his view the large cyclical power of these basic factors and their changing combination creates a presumption that long cycles will be produced through their joint action; and that the more readily if there be long cycles in the increases of the national population. Errors of estimate may tend to reinforce variability of population increase and uncertainty may be held to be a strategic factor in the prolongation of the construction cycles.[11] Thus a certain rational approach has been made towards explaining some characteristics of the long cycles in residential construction, but no sufficient reasons have been supplied in this analysis for the timing and the regularity in the course of the cycles.

The somewhat sweeping statements which have been advanced with respect to the lack of co-ordination between the long building

[8] W. H. Newman, *op. cit.*, p. 18.

[9] C. F. Roos, *Dynamic Economics*, Bloomington, Indiana, 1934, p. 69. Of the ninety-nine industries examined by Burns, building permits manifested cycles at least twice as wide as those of seventy and at least three times as wide as those of forty (A. F. Burns, *Production Trends since 1870*, pp. 230 ff.).

[10] W. H. Newman, *op. cit.*, p. 47.

[11] A. F. Burns, "Long Cycles in Residential Construction," *Economic Essays in Honor of W. C. Mitchell*, N.Y., 1935, pp. 83 ff.

cycles and business fluctuations have recently been revised by Clarence D. Long, Jr. on the basis of careful comparisons made between the cycles of both types.[12] This comparison goes to show that periods of severe business depressions are also periods of severe depression in residential building; that in the case of severe depressions building displays an unmistakable tendency to precede general business on the downturn, and that after a severe depression building recovery appears to lag considerably behind recovery in general business, not so much in the timing of the initial revival as in the degree of subsequent recovery. In Long's view the explanation of the lead of building on the downturn is to be found in the acceleration principle in terms of which the volume of building tends to fluctuate, not with movements in the total income or total consumption demand, but rather with the movements in the increase of these factors. This consideration appears to lead to the conclusion that also resumption of building—as is actually the case in Europe—should take place before the general business revival. In Long's view, however, building does not resume until some at least of the following conditions have been satisfied: the excess supply of buildings provided by the preceding boom must be used up; a decline in costs of construction and land prices must take place; a decrease in the going rate of interest must come about since the capitalized value of buildings will be the higher, the lower is the rate of interest; finally an improvement in public confidence in the future must occur.[13] In his subsequent explanation of the severity of the building cycles Long emphasizes, apart from the acceleration principle, the lag in the production process, the fact that residential buildings are increasingly supplied by ready-made production for a future unknown market, that building industry shows a high proportion of direct costs and rigidity of these costs, and that the computation of the rate of profit to be obtained from investments in residential buildings is based on many irrational criteria, with the result that this rate becomes really another word for confidence.

Whatever may be the merits of this analysis with respect to the explanation of the severity of building fluctuations it contributes little, if anything, towards the explanation of the duration and timing of such fluctuations. Long considers the lag in the production process as a primary factor in determining the minimum length of the building cycle.[14] His arguments are that after years of depression in building the supply of contractors and skilled workers is greatly depleted and

[12] Clarence D. Long, Jr., "Long Cycles in the Building Industry," *The Quarterly Journal of Economics*, Vol. 53, May, 1939, pp. 371 ff.

[13] Long, *op. cit.*, pp. 383, 384.

[14] Long, *op. cit.*, p. 390.

that time is also required to build up the industries supplying materials and machinery, that, as business expands, it must bid against other consumption-goods industries for the (diminishing) capital and labor available; but any such arguments are equally applicable to other industries whose activity is not marked by abnormal cyclical movements. In addition it is to be emphasized that the European building industry does not differ significantly, as to its organization and methods of construction, from its American counterpart, and that nevertheless the European building cycles, which are equally marked by considerable though not excessive amplitudes, do not show any abnormal duration.

As distinct from the attempts so far made to explain the main characteristics of the American building cycles, the question may be raised whether and to what an extent the salient features in the development of American building activity can be analyzed in the light of the experience which has been gained from the observation of corresponding European processes.

Offhand it could be anticipated that also in this country changes in the development of the absolute urban ground rent might exert a strong influence upon the course of building activity. If this course significantly differs from that evidenced by the development of European building activity, the difference may primarily be explained in terms of a different behavior of the urban ground rent. Hence the factors are to be analyzed which are primarily responsible for this behavior. The urban ground rent may show a different behavior on account of the fact that the relations between changing yields of the buildings and changing costs of construction vary in a way which differs from that evidenced by European experience. But, moreover, it may be that there exist factors which tend to prevent the ground rent from exercising its full influence upon the development of building activity. In both respects some recent studies have been highly instructive and illuminating.

As is the case in Europe the urban ground rent can be assumed to increase when recovery takes its start after the low of the business depression has been overcome, if at this juncture demand for housing facilities exceeds the existing supply owing to the cessation of building activity in the preceding depression period. If this relation between demand and supply obtains, rentals are likely to rise and, at relatively low costs of construction and depressed operating costs, the prospective net yield of new buildings can be expected to be fairly high.

The incentive to building activity which under such circumstances results from the increasing ground rent, has been statistically analyzed by C. F. Roos in the sixth chapter of his book, *Dynamic Economics*,[15]

[15] Bloomington, Indiana, 1934.

in which he has carefully compared the time series of various indices whose mutual behavior tends to explain the fluctuations of building activity in the city of St. Louis over a period of more than 30 years. According to his findings a ten-per-cent net rental income on buildings based on the replacement costs usually leads to a high level of construction activity—a five-per-cent return to a very low level. In St. Louis the former situation occurred in 1903 and again in 1922. It may be mentioned, however, that the net yield of the typical family building unit, on which this statistical analysis is based, does not represent the urban ground rent strictly speaking since the costs of the lot are included among the replacement costs used in the calculations; the replacement costs may therefore be inflated by capitalized portions of the ground rent. But what matters here is less the development of the actual ground rent than its changes and these are well reflected in the fluctuations of the net rental yield.

The clear result of this analysis is to the effect that, as in Europe, fluctuations of the absolute ground rent are brought about by disproportionate developments of the monetary factors which are instrumental in producing the ground rent; these factors are gross rentals on the one hand, costs of construction and operating costs on the other. Since disproportionate movements of these factors are primarily brought about by general business fluctuations, the assumption appears to be justified that also in this country the forces which are generating business fluctuations play a role in producing cyclical movements of residential construction.

Why did it happen that such conditions as tend to provide strong incentives to brisk building activity presented themselves only twice in St. Louis in the course of more than thirty years in spite of the fact that four major business cycles developed during that period? The picture is no less striking in Chicago where, according to the analysis of Homer Hoyt,[16] no outright building boom took place between 1897 and 1917. The most plausible explanation of these phenomena, which are reflected in the fluctuating figures of building activity compiled for the country as a whole, is to the effect that in most cities of this country residential building, when once briskly under way in prosperity periods, does not decline perceptibly when the ground rent starts declining, both because of considerable reductions in rentals due to oversupply of dwelling facilities and because of increases in the costs of construction which are frequently due to rising wages. Significant reductions of net returns would appear to indicate a situation which is hardly fit for providing incentives to additional building activity in excess of such a volume of dwellings as would be needed for providing

[16] Homer Hoyt, *op. cit.*

housing facilities for the normal annual increases in families. Nevertheless in St. Louis building activity, which had started to rise in 1904–05 when the net rental yield of a typical family building unit was about 10 per cent, did not decline perceptibly until after 1909 when the net yield was about 7 per cent. The next building boom which started in 1922, when again a 10-per-cent yield had been reached, continued through 1928 although the net yield was steadily declining after 1924. In addition we are probably safe in assuming that the net yields as shown by the calculations are hardly representative of the actual net yield earned by those who participated in building during the later stages of the boom when intensified speculation in suburban lots had resulted in heavily inflated land values whose level was not even justified by the belief in a lasting prosperity. Hence, the net yield of almost all houses built for some time before the building peak had been reached and during the entire period following upon the peak was probably low from the outset and was likely to be swept away through relatively moderate reductions in rentals.

Hence, as compared with corresponding European developments, the American building booms are marked by the fact that even considerable general declines of the urban ground rent do not result in checking building activity when it is once under way. Such situations could hardly develop but for the structure of the American mortgage market which differs from the corresponding European structure in so far as a large variety of credit institutions participate in financing residential construction, among others also commercial banks. H. Hoyt gives a picture of the various ways in which state and national banks have contributed towards financing real-estate transactions during the Chicago building boom of the twenties.[17] The amazing elasticity of credit expansion which has been an important factor in stimulating building booms is not paralleled in any European country where capital is withdrawn from large-scale investment in residential construction when the ground rent starts declining under the pressure of increasing costs of construction and when, almost simultaneously, the mortgage market tightens since its capacity to expand is held within relatively narrow limits.

It is evident from this analysis that speculative tendencies fostered by the conditions of the mortgage market result in impairing the control of the absolute ground rent over building activity when the latter is rapidly expanding in prosperity periods. It may thus happen, as it actually happened during the last building cycle, that in the course of the ensuing depression rentals are forced down to the level which had obtained before the boom had taken its start. Hence, in spite of de-

[17] Homer Hoyt, *op. cit.*, pp. 445 ff.

creasing costs of construction, any general increases in the ground rent which had been gained in the course of the prosperity period may be swept away and the phenomenon of the absolute ground rent which is a characteristic feature of the European situation can hardly be said to occupy the same place in the far more dynamic course of American developments.

The oversupply of housing facilities which has been created through the building boom may be so considerable that it prevents rentals from rising sufficiently during the subsequent recovery period to re-establish such a relation between the returns on houses and reduced costs of construction as would be necessary for stimulating building activity.[18] Thus the following business cycle may take its course without being accompanied by any remarkable building activity.

Moreover, at this juncture, even general increases in the urban ground rent may be prevented from exerting their influence upon building activity by a factor whose importance has equally been demonstrated in the analysis of the St. Louis situation made by C. F. Roos: the large volume of properties, which, owing to preceding foreclosures, are depressing the real-estate market, increasing the supply of buildings offered for sale, affecting the general value of buildings in the community, impairing the equities and mortgages, and paralyzing the willingness of lending institutions and investors to grant loans needed to stimulate residential construction. The depressed state of the real-estate market which results from this situation may be an additional reason why an entire business cycle may pass without giving rise to substantial revival of building activity. As a rule, even the first years of the next recovery may be needed for clearing the situation and this phenomenon may account for the fact that, as distinct from the corresponding European conditions, the influence of the ground rent is not strong enough in this country to stimulate building activity during the earlier stages of general recovery. This factor was strongly emphasized in the last (sixth) Annual Report of the Federal Home Loan Bank Board.[19] The fact that rentals were on relatively so much lower levels than building costs during the past five or six years was rendered responsible for the limited volume of private building in that period. However, although a distinctly improving rent-cost relationship was found to be under way, a large "overhang"

[18] In an examination made of a number of New York City cost factors, including tax rates, levies, assessments, building-material prices, wage rates, and new-mortgage interest rates, for periods beginning as early as 1868, it was found that in the case of nearly all the cost factors, the declines showed slight and lagging response to fluctuations in building. In no case was a decline as great as the previous rise (C. D. Long, *op. cit.*, p. 397).

[19] For the period July 1, 1937 to June 30, 1938, p. 5.

of properties repossessed by financial institutions during the depression was said to have exerted continuous pressure on the building market since under these conditions existing homes could be bought at less than reproduction costs. The reduction of the overhang estimated at more than 900,000 nonfarm dwelling units in one- to four-family houses was expected to determine in part the time and extent of any further revival of building activity.

If this picture is on the whole correct, it does not appear necessary to assume, as is frequently done, that the forces which are controlling the American building cycles differ fundamentally from those which are responsible for general business fluctuations. But we have to assume that these forces, when operating in the field of residential construction, are driven to much wider extremes than in other lines of productive activity. The oversupply in dwelling units created during building booms is as a rule considerable even when measured in terms of such a demand as would correspond to conditions of lasting prosperity, whereas the demand is, in fact, fairly elastic and exposed to heavy reductions by alterations in migratory movements, doubling-up processes, and the like. The tendency toward establishing disequilibria between demand and supply is partly due to the fact that highly speculative motives are an important element in real-estate transactions which in boom periods are largely actuated by exaggerated expectations pinned on gains to be reaped from future "unearned increments." The sluggishness of land values in responding even to considerable declines of the ground rent accounts for the fact that crises in the real-estate market, as a rule, take place many months or even several years after the general decline of business activity had taken its start. During the last Chicago building boom the real-estate market had begun to grow dull as early as 1927, but asking prices for properties continued to advance slowly even in 1927 and 1928. The collapse of the peak values of 1928 came only in June, 1931, almost two years after business prosperity had been reversed.[20]

Speculation in real-estate properties could hardly assume such features as to ignore almost completely the importance of the urban ground rent as a controlling factor of building activity, unless the institutional setup of the American mortgage market should include banks provided with great capacity for credit expansion and willing to make the largest possible use of this capacity. Under these conditions overexpansion of credit facilities in boom periods beyond any reasonable needs of real-estate trading and constructional activity was bound to develop. This overexpansion was paralleled by exaggerated credit contraction in ensuing depression periods with the result that the

[20] H. Hoyt, *op. cit.*, pp. 265 ff.

downward spirals of deflation gave rise to rapidly accumulating foreclosures. Thus for prolonged periods during which the real-estate market was depressed through large volumes of institutional real-estate holdings resulting from foreclosures, the ground rent was again prevented from exerting its regulatory influence upon building activity, even though the demand for dwellings increasingly exceeded the existing supply. It thus appears that in the course of the American building cycles there were only certain periods—the early stages of building booms and depression periods—during which building activity was effectively controlled by the development of the ground rent. The forces which prevented the ground rent from exercising this control appear to have been instrumental also in determining the duration and the timing of the building cycles.

It is not within the scope of the present paper to discuss problems of housing policy. But any inferences which, with respect to such policy, can be drawn from the preceding analysis are likely to center on the problem of how to secure the development of the urban ground rent a far greater influence upon building activity than has been the case in previous periods. Important steps in this direction have been taken by the creation and functioning of Federal housing agencies, such as the Federal Housing Administration and the Home Loan Bank Board which have endeavored to establish sound appraisal systems for mortgaging purposes and to improve the organization of the mortgage market with a view to safeguarding it against disastrous foreclosure situations. But so far little has been done for preventing future building booms from assuming such proportions that the housing market would again be inflated by oversupply of marketable dwellings. It can even be suggested that the capacity of the real-estate credit system to expand in periods of brisk building activity has been increased by such measures as federal mortgage insurance; similarly the participation of commercial banks in financing real-estate transactions and constructional activity, though better regulated than before, has been stimulated rather than discouraged. Hence, some of the fundamental problems involved in the particular features of the American building cycles appear to have received relatively little attention. Apart from the important tasks which housing policy is expected to fulfil particularly in the field of providing adequate housing facilities for the large masses of low-income families, one of its major objectives appears to be to adopt such measures as might result in adjusting the fluctuations of building activity more closely to the European pattern and in making the development of the absolute ground rent a force of paramount influence upon these fluctuations.

Social Security Board
Washington, D. C.

THE DEGREE OF DAMPING IN BUSINESS CYCLES

By T. Koopmans

1. STATEMENT OF THE PROBLEM

In recent years a number of theoretical and statistical investigations[1] in the field of business-cycle analysis have made use of difference equations, or mixed difference and differential equations. These equations serve to study the possible endogenous movements of a schematized economic system governed by a set of as many "structural" relations as there are variables, the movements of which are considered. In the investigations referred to, the treatment of such a set of equations has been to eliminate successively all variables but one, which leaves one "final equation" from which the possible movements of the system under consideration are studied. If the final equation in the variable Z_t is a linear homogeneous difference equation or mixed difference and differential equation, possible movements of Z_t are found by substituting for it in the final equation an expression of the form[2]

$$Z_t = \mathcal{R}Ae^{zt} = |A|\, e^{xt} \cos y\left(t + \frac{\alpha}{y}\right), \tag{1}$$

where

$$A = |A|\, e^{i\alpha} \tag{2}$$

and

$$z = x + iy \tag{3}$$

[1] R. Frisch, "Propagation Problems and Impulse Problems in Dynamic Economics," *Economic Essays in Honour of Gustav Cassel*, reprinted as Publ. 3, University Institute of Economics, Oslo.

R. Frisch and H. Holme, "The Characteristic Solutions of a Mixed Difference and Differential Equation Occurring in Economic Dynamics," Econometrica, Vol. 3, 1935, p. 225.

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J. Tinbergen, "Ein Schiffsbauzyklus?" *Weltw. Archiv*, 1931, p. 152; *An Econometric Approach to Business Cycle Problems*, Paris, 1937; *Les Fondements mathématiques de la stabilisation du mouvement des affaires*, Paris, 1938; *Business Cycles in the U.S.A., 1919–1932*, League of Nations, 1939; and other papers.

[2] Z_t is taken as representing the *deviation from average or from trend* of some economic quantity.

are complex numbers, and the symbol $\mathcal{R}$ means "the real part of." This leads to an equation in z, called the characteristic equation, to each root of which corresponds a solution (1) of the final equation (with an arbitrary value of A), i.e., a possible type of movement of the system. If a root is complex ($y \neq 0$), the corresponding solution Z_t represents an oscillation with a *period*

$$T = \frac{2\pi}{y} \tag{4}$$

and a gradually changing amplitude. To a real root corresponds a one-sided movement of the "exponential" type. In the former case the amplitude, in the latter case the value of Z_t, increases or decreases in the constant ratio

$$D = 1 : e^{x} \tag{5}$$

per unit of time, which may be called the *damping ratio* of a solution.

The adequacy of the above mathematical apparatus in dealing with problems of business-cycle analysis will not be discussed here. Granted that it has some relevance as a first approximation to reality, it will be clear that an economic system will be more stable, and less affected by disturbances from outside, according as the damping ratios of the movements satisfying the equations by which it is described are higher. It is, then, a matter of theoretical and practical importance to investigate how the value of D for any particular solution depends on the various coefficients in the structural relations. The effect of changes to certain structural coefficients on the coefficients in the final equation are easily established in any particular case. It is the purpose of this note to derive a few propositions about the dependence of the damping ratio and the period of any solution on the coefficients occurring in the final equation.

2. DAMPING RATIO AND PERIOD OF A SOLUTION OF A DIFFERENCE EQUATION DIFFERENTIATED WITH RESPECT TO THE COEFFICIENTS OF THE EQUATION

If the final equation has the simple form of a difference equation

$$A_N Z_t + A_{N-1} Z_{t-1} + \cdots + A_n Z_{t-N+n} + \cdots + A_0 Z_{t-N} = 0, \tag{6}$$

the characteristic equation is an algebraic equation of degree N

$$F(w) \equiv A_N w^N + A_{N-1} w^{N-1} + \cdots + A_0 = 0 \tag{7}$$

in the unknown

$$w = e^{z}. \tag{8}$$

To any root w of this equation corresponds, mathematically, an infinite sequence of solutions of (6) of the form (1), all with the same damping ratio, but with different periods which corresponds to the different arguments y that can be attached to a complex number w. Figure 1 shows lines corresponding to constant values of damping ratio and period in the plane of w, the values of the period indicated being the largest corresponding to any given value of w. In what follows we shall confine our attention to the solutions with the largest period.

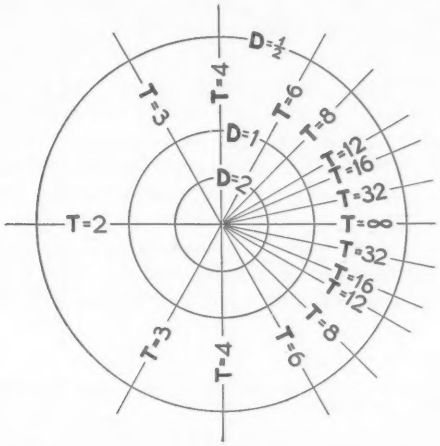

FIGURE 1.—Lines of constant damping ratio (D) and period (T) in the plane of the complex number w.

Let w be a single root of (7), and δA_n a small and real variation of the real coefficient A_n of the term $A_n Z_{t-N+n}$ in (6). The constant $-N+n$ in the expression $t-N+n$ for the time point to which this term refers will be called the *timing*, or *time lag*, of that term (i.e., with respect to the leading term $A_N Z_t$). The variation δw in w corresponding to δA_n satisfies

$$F(w + \delta w) + \delta A_n \cdot w^n = 0, \tag{9}$$

or, since

$$F(w + \delta w) = F(w) + F'(w)\delta w = F'(w)\delta w, \tag{10}$$

$$\delta w = -\frac{w^n}{F'(w)} \cdot \delta A_n. \tag{11}$$

The resulting variation in D is, according to (5), found from

$$\delta D = \delta e^{-z} = -e^{-z}\delta x = -D\Re\delta z, \tag{12}$$

where, owing to (8),

$$\delta z = \delta \log w = \frac{\delta w}{w}. \tag{13}$$

Combining (12), (13), and (11), we obtain

$$\frac{\partial D}{\partial A_n} = D\Re\frac{w^{n-1}}{F'(w)} = D\Re\frac{e^{(n-1)z}}{F'(e^z)}. \tag{14}$$

Considered as a function of n taken as a continuous variable, this is an expression of the type (1). Therefore, *if the derivatives of the damping ratio of any solution of the final equation with respect to the coefficients of that equation are plotted against the timing of the terms with the corresponding coefficients, a set of points is obtained which lie on a curve having the shape of the solution under consideration,* provided that this solution corresponds to a single root of the characteristic equation. The curve so obtained may be called the *damping-ratio variation curve.*

For purposes of numerical computation it is convenient to put (14) in the form

$$\frac{\partial D}{\partial A_n} = \frac{D^{-n+2}}{|F'(w)|}\cos\left[(n-1)y - \arg F'(w)\right], \tag{15}$$

computing modulus and argument of $F'(w)$ from the moduli and arguments of the differences between w and each of the other roots[3] of (7), which are usually determined together with w. If, in particular, z is real, so $y=0$, $F'(w)$ is also real, and (15) reduces to

$$\frac{\partial D}{\partial A_n} = \frac{D^{-n+2}}{F'(w)} \tag{16}$$

for positive as well as for negative values of $F'(w)$.

A similar proposition holds, it will be clear, for the derivatives of the period T of any oscillatory solution. From (4) and (8) we have

$$\delta T = -\frac{2\pi}{y^2}\delta y = -\frac{T^2}{2\pi}\Im\delta z, \tag{17}$$

like

if $\Im$ followed by a complex number denotes the imaginary part of that number apart from the factor i.[4] By (13) and (11), again, this leads to

[3] See equation (22) below.

[4] So $z=\Re z+i\Im z$.

$$(18)\quad \frac{\partial T}{\partial A_n} = \frac{T^2}{2\pi}\mathcal{I}\frac{w^{n-1}}{F'(w)} = \frac{T^2 D^{-n+1}}{2\pi\,|\,F'(w)\,|}\sin\,[(n-1)y - \arg F'(w)],$$

an expression which reduces to 0 if $y=0$.

The curve obtained by taking n in the right-hand member of (18) as a continuous variable represents again a solution of (6) corresponding to the single root w of (7) and may be called the *period variation curve.* Upon comparing (18) and (15) it appears that *the period variation curve lags one-quarter of the period T behind the damping-ratio variation curve.*

3. TINBERGEN'S EQUATION FOR THE UNITED STATES AS AN EXAMPLE

As an illustration, the above formulae may be applied to the equation

$$(19)\quad Z_t - 0.398Z_{t-1} + 0.220Z_{t-2} - 0.013Z_{t-3} - 0.027Z_{t-4} = 0,$$

used by Tinbergen[5] in describing the cyclical mechanism inherent in the United States economy of the period studied by him. The roots of the corresponding characteristic equation are

$$(20)\qquad w_{1,2} = 0.131 \pm 0.512i, \qquad w_3 = 0.385, \qquad w_4 = -0.248,$$

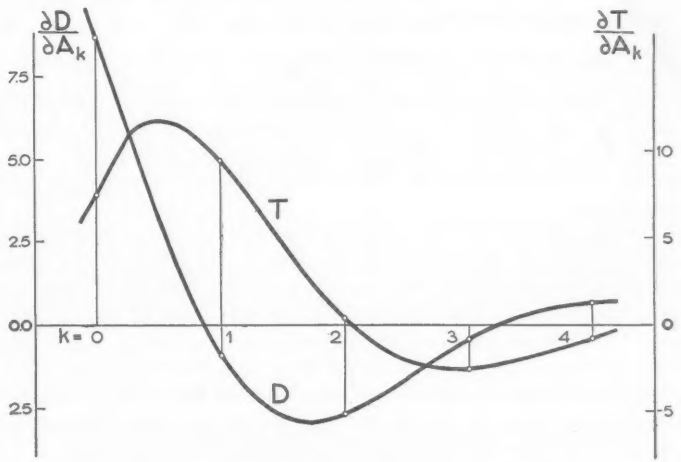

FIGURE 2.—The damping-ratio variation curve (D) and the period variation curve (T).

[5] J. Tinbergen, *Business Cycles in the United States, 1919–1932,* League of Nations, 1939, p. 140.

while for the solution corresponding to the root w_1

$$D = 1.89, \qquad T = 4.76, \tag{21}$$

the time unit being one year.

The values of $\partial D/\partial A_n$ and $\partial T/\partial A_n$ for this solution are given in Table 1, the corresponding curves in Figure 2. Using the figures in Table 1, Tinbergen computed[6] the effect on damping ratio and period

TABLE 1

DERIVATIVES OF THE DAMPING RATIO D AND THE PERIOD T OF THE OSCILLATORY SOLUTION OF EQUATION (19) WITH RESPECT TO THE COEFFICIENTS A_n OF THAT EQUATION

n	A_n	$\partial D/\partial A_n$	$\partial T/\partial A_n$
0	−0.027	8.71	7.55
1	−0.013	−0.89	9.51
2	0.220	−2.68	0.38
3	−0.398	−0.45	−2.57
4	1.000	0.64	−0.78

of given changes in certain structural coefficients, e.g., in the marginal propensity to consume, in the coefficient connecting movements in profits with movements in investment, in the percentage of an increase in profits distributed as dividends, etc.

4. SOME GENERAL CONCLUSIONS FROM THE FORMULAE DERIVED

The damping-ratio and period variation curves provide a means by which in any particular case economic conclusions about the effect of changes of structural coefficients on the possible endogenous movements of the system may be derived. Some general inferences may be stated now.

Owing to the oscillatory nature of the variation curves corresponding to oscillatory movements of the system, the effect of any given change of the coefficient of a term in the final equation on damping ratio or period of such movements depends very much on the timing of that term. Equal additions to coefficients in consecutive terms may even have opposite effects on the damping ratio, or on the period. Further, if the timing of a given term happens to be such that the effect of a certain variation of its coefficient on the damping ratio is almost nil, the effect on the period is close to a maximum, and vice versa.

Apart from the alternation of the effects on damping ratio and period, there is a tendency for the effect of a given variation of a coefficient on one or the other of these to *increase* with increasing time

[6] *Loc. cit.*, p. 173.

lag of the term concerned if the solution studied is a *damped* oscillation, to *decrease* if it is an *expansive* oscillation. This is due to the factor D^{-n} in (15) and (18). Accordingly, if the system already admits only of damped movements, policy affecting structural coefficients that measure causal connections acting with a large time lag will in general offer the best chances for a further increase of the damping ratio—unless its effect happens to be reduced by the proximity of a zero point in the damping-ratio variation curve.[7]

The amplitude factors and phase constants of the two variation curves appear to be wholly determined by the complex number $F'(w)$, i.e., by the position of the root $w=w_1$ under consideration, with respect to the remaining roots $w_2, w_3, \cdots$ of the characteristic equation. For

$$F'(w_1) = (w_1 - w_2)(w_1 - w_3) \cdots (w_1 - w_N). \tag{22}$$

As $|F'(w)|$ occurs in the denominators in (15) and (18), proximity of another root w_2 to the root w_1 considered will lead to a large effect of changes in the coefficients on damping ratio and period. An interesting special case is that in which w_2 is the complex conjugate w_1^* of w_1. Since for solutions with a very long period w_1 and w_1^* are close to each other (even if the damping ratio is only moderate), the period and damping ratio of such solutions are highly sensitive to changes in the structural coefficients.

In the case in which the final equation has only three consecutive terms and admits of an oscillatory solution, w and w^* are the only roots, and $F'(w)$ has the argument $\pi/2$, whence $\partial D/\partial A_1=0$. In the case of equation (19) the small terms with Z_{t-3} and Z_{t-4} introduce further roots w_3, w_4, which are sufficiently small to let the timing of the zero point of the damping-ratio variation curve still fall between that of the terms Z_t and Z_{t-2}.

According to (16) the damping ratio of a nonoscillatory solution is affected in the same direction by changes in each of the coefficients of the final equation. The sign of $F'(w)$ is in this case governed by the following rule. If $w=w_1$ is a real root of (7), any complex root w_2 will, together with its conjugate w_2^*, contribute to $F'(w_1)$ a positive real factor

$$(w_1 - w_2)(w_1 - w_2^*) = (w_1 - \mathcal{R}w_2)^2 + (\mathcal{J}w_2)^2. \tag{23}$$

Therefore, if we arrange the real roots in decreasing order, the sign of

[7] A certain variation of a structural coefficient may, of course, lead to changes in several coefficients in the final equation. In that case, the effects of each of these changes may, as long as small variations are considered, be added to give the total effect, and the above argument is easily generalized so as to cover such cases also.

$F'(w_1)$ will, according to (22), be positive if w_1 has an odd-numbered place in this order, negative if its place is even-numbered. Thus, the damping ratio of the least damped nonoscillatory solution is increased by an increase in any coefficient of the final equation, that of the next is diminished, and so forth.

5. EFFECTS OF THE ADDITION OF SMALL TERMS AND EFFECTS OF SMALL CHANGES IN TIME LAGS ON THE DAMPING RATIO AND PERIOD OF THE SOLUTIONS OF A MORE GENERAL TYPE OF EQUATION

The above variation calculus is easily extended to cases in which the final equation contains derivatives with respect to time, or integrals over time, and to cases in which the time lags of its terms are not entire multiples of a common unit. It may also be used to study the effect of a small change in the timing of a term, or of a group of terms.

As soon as the characteristic equation is no longer algebraic, it is better to have it in terms of z than in terms of w. For in such cases a root w generally does not solve the equation for all possible values of its argument, but only for one of them. The use of w would then require that for any root w the argument be specified, and just this is implied in the use of z, according to (8).

The final equation may quite generally be written

$$\mathcal{L}Z_t = 0, \tag{24}$$

where $\mathcal{L}$ denotes some operation applied to the time function Z_t. The only properties which we require of the *operator* $\mathcal{L}$ are:

(a) that it be real, i.e., that $\mathcal{L}Z_t$ be real if Z_t be real;

(b) that it be linear, i.e., that for any two complex numbers α_1 and α_2

$$\mathcal{L}(\alpha_1 Z_t^{(1)} + \alpha_2 Z_t^{(2)}) = \alpha_1 \mathcal{L}Z_t^{(1)} + \alpha_2 \mathcal{L}Z_t^{(2)}, \tag{25}$$

(c) that, for any value of z,

$$\mathcal{L}e^{zt} = L(z) \cdot e^{zt}, \tag{26}$$

where $L(z)$ is a differentiable function of the complex variable z, which we shall call the *operator function* of $\mathcal{L}$.

As all these properties are satisfied in the applications quoted above, the question may here be disregarded whether, or under what conditions, (c) is a consequence of (b).

The characteristic equation now runs

$$L(z) = 0. \tag{27}$$

We shall first study the effect of the addition to $\mathcal{L}$ of a small operator $\delta\mathcal{L}$ satisfying the same properties as $\mathcal{L}$. If an increase in the coefficients

of some terms in $\mathcal{L}$ is to be studied, $\delta\mathcal{L}$ will be chosen so as to combine the additions to those terms. It may however also be chosen so as to investigate the effect of the addition of small new terms in $\mathcal{L}$.

The first variation of the characteristic equation runs, if z is a single root of (27),

$$L'(z)\delta z + \delta L(z) = 0, \tag{28}$$

where $\delta L(z)$ is the operator function of $\delta\mathcal{L}$. Again using (12), we find

$$\delta D = D\mathcal{R}\frac{\delta L(z)}{L'(z)} = \left[\delta\mathcal{L}\mathcal{R}\,\frac{D}{L'(z)}\,e^{zt}\right]_{t=0}, \tag{29}$$

as $\mathcal{R}$ and $\delta\mathcal{L}$ are interchangeable on account of the above properties (a) and (b). Similarly, from (17),

$$\delta T = \frac{T^2}{2\pi}\left[\delta\mathcal{L}\mathcal{J}\,\frac{e^{zt}}{L'(z)}\right]_{t=0} = \left[\delta\mathcal{L}\mathcal{R}\,\frac{T^2}{2\pi i L'(z)}\,e^{zt}\right]_{t=0}. \tag{30}$$

It appears from (29) and (30) that *the effects, on damping ratio and period of a solution of* (24) *corresponding to a single root z of* (27), *of the addition of a small operator* $\delta\mathcal{L}$ *to* $\mathcal{L}$ *are obtained by applying that operator to the functions*

$$Z_t^{(D)} = \mathcal{R}\,\frac{D}{L'(z)}\,e^{zt} \tag{31}$$

and

$$Z_t^{(T)} = \mathcal{R}\,\frac{T^2}{2\pi i L'(z)}\,e^{zt} \tag{32}$$

respectively and substituting $t=0$. *These functions are represented by the damping-ratio variation curve and the period variation curve, and form solutions of* (24) *corresponding to the root z of* (27) *considered, with specific complex amplitude factors which make* (32) *lag one-quarter of the period behind* (31).

A special case of variation of the final equation is that by which the time lag of a term, or of a group of terms is changed. Combining into the operator $\mathcal{M}$ those terms of $\mathcal{L}$ which are all subjected to the same small change $\delta\theta$ in timing, we may define $\delta\mathcal{L}$ by

$$\delta\mathcal{L}Z_t = \mathcal{M}Z_{t+\delta\theta} - \mathcal{M}Z_t. \tag{33}$$

As, for variations of the first order,

$$\mathcal{M}(e^{z(t+\delta\theta)} - e^{zt}) = \mathcal{M}(e^{z\delta\theta} - 1)e^{zt} = \delta\theta\cdot\mathcal{M}ze^{zt}, \tag{34}$$

the effect on damping ratio and period is found by applying $\delta\theta \cdot \mathcal{M}$ to the functions

$$Z_t^{(D,\text{tim})} = \mathcal{R} \frac{Dz}{L'(z)} e^{zt}, \tag{35}$$

and

$$Z_t^{(T,\text{tim})} = \mathcal{R} \frac{T^2 z}{2\pi i L'(z)} e^{zt}, \tag{36}$$

and taking $t=0$. The curves represented by these functions may be called the *timing variation curves* relating to damping ratio and period.

6. APPLICATION TO AN EQUATION USED BY KALECKI

To demonstrate the working of the above formulae, these may be applied to the equation

$$\mathcal{L}Z_t \equiv \frac{dZ_t}{dt} - aZ_t + cZ_{t-\theta} = 0, \tag{37}$$

introduced by Kalecki[8] and also discussed by other writers.[9] The characteristic equation corresponding to (37) runs, in terms of z,

$$L(z) \equiv z - a + ce^{-z\theta} = 0 \tag{38}$$

whence

$$L'(z) = 1 - \theta ce^{-z\theta}. \tag{39}$$

The derivatives of damping ratio and period of any solution of equation (37) with respect to a, c, and θ are found by inserting for $\delta\mathcal{L}$ in (29) and (30) the small operators defined by

$$\delta\mathcal{L}_a Z_t = -\delta a \cdot Z_t, \quad \delta\mathcal{L}_c Z_t = \delta c \cdot Z_{t-\theta}, \quad \delta\mathcal{L}_\theta Z_t = cz\delta\theta \cdot Z_{t-\theta}, \tag{40}$$

respectively. If applied to e^{zt}, these operators yield functions of t which for $t=0$ assume the values

$$-\delta a, \qquad \delta c \cdot e^{-z\theta}, \qquad cz\delta\theta \cdot e^{-z\theta}. \tag{41}$$

Using (39), we find, therefore,

[8] M. Kalecki, "A Macrodynamic Theory of Business Cycles," ECONOMETRICA, Vol. 3, 1935, p. 327.

[9] See the papers by Frisch and Holme, and by James and Belz, quoted above.

$$\frac{\partial D}{\partial a} = -\Re\frac{D}{1-\theta c e^{-z\theta}}, \qquad \frac{\partial D}{\partial c} = \Re\frac{D}{e^{z\theta}-\theta c},$$

$$\frac{\partial D}{\partial \theta} = \Re\frac{Dcz}{e^{z\theta}-\theta c}, \tag{42}$$

and similar expressions for the derivatives of T.

Numerical values of the derivatives for any particular solution are easily calculated from these formulae.

League of Nations
Geneva

OFFICERS AND COUNCIL FOR 1940

The Council of the Econometric Society has elected the following officers for 1940: Joseph A. Schumpeter, President; J. Tinbergen, Vice-President; and Alfred Cowles, Secretary and Treasurer.

The Fellows of the Society have elected the following Council members for terms to expire in December, 1942: Constantino Bresciani-Turroni, Irving Fisher, Charles F. Roos, and Joseph A. Schumpeter; and for a term to expire in December, 1941: J. Tinbergen (succeeding Wladyslaw Zawadzki, deceased). Other Council members whose terms expire in December, 1941, are: Albert Aupetit and Arthur L. Bowley. Those whose terms expire in December, 1940, are: Ragnar Frisch, John Maynard Keynes, and F. Zeuthen. Alfred Cowles, because of his position as Secretary and Treasurer, is an ex officio member of the Council.

ALFRED COWLES
Secretary

The University of Chicago

ELECTION OF FELLOWS

The Fellows of the Econometric Society have elected the eight new Fellows, whose names and partial bibliographies follow:

WILLIAM LEONARD CRUM, Harvard University, Cambridge, Massachusetts, U. S. A.

"Cycles of Rates on Commercial Paper," *Review of Economic Statistics*, Vol. 5, January, 1923, pp. 17–27.

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Also numerous papers in various journals, especially in the *Review of Economic Statistics* and the *Harvard Business Review.*

HAROLD THAYER DAVIS, Northwestern University and Cowles Commission for Research in Economics, Evanston, Illinois, U. S. A.

(With others) *A Debate on the Theory of Relativity.* Chicago, 1927.

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T. KOOPMANS, League of Nations, Geneva, Switzerland.

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SIMON S. KUZNETS, University of Pennsylvania, Philadelphia, Pennsylvania, U. S. A.

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FREDERICK CECIL MILLS, Columbia University, New York, New York, U. S. A.

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GUNNAR MYRDAL, Stockholms Högskola, Stockholm, Sweden.

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GERHARD TINTNER, Iowa State College, Ames, Iowa, U. S. A.

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FELLOWS OF THE ECONOMETRIC SOCIETY

JANUARY, 1940

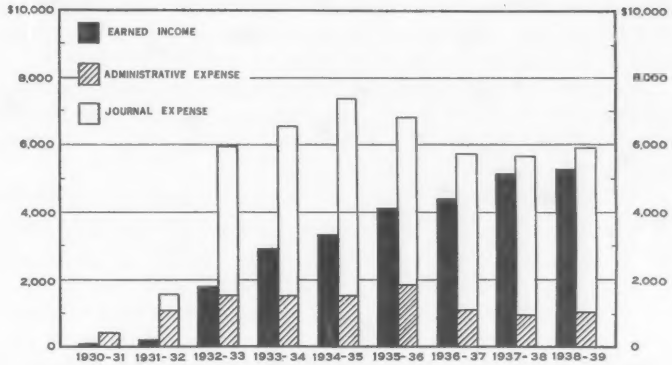
Mr. R. G. D. ALLEN, London, England.
Professor LUIGI AMOROSO, Rome, Italy.
Professor OSKAR N. ANDERSON, Sofia, Bulgaria.
Dr. ALBERT AUPETIT, Paris, France.
*Professor P. BONINSEGNI, Lausanne, Switzerland.
Professor ARTHUR L. BOWLEY, London, England.
Professor COSTANTINO BRESCIANI-TURRONI, Giza, Egypt.

* Deceased since publication of list in ECONOMETRICA, Vol. 7, July, 1939, pp. 286–287.

Mr. Alfred Cowles, Chicago, Illinois, U.S.A.
Professor William Leonard Crum, Cambridge, Massachusetts, U.S.A.
Professor Harold T. Davis, Evanston, Illinois, U.S.A.
Professor Gustavo Del Vecchio, Bologna, Italy.
Professor François Divisia, Paris, France.
Professor Griffith C. Evans, Berkeley, California, U.S.A.
Dr. Mordecai Ezekiel, Washington, District of Columbia, U.S.A.
Professor Irving Fisher, New Haven, Connecticut, U.S.A.
Professor Ragnar Frisch, Oslo, Norway.
Professor Corrado Gini, Rome, Italy.
Professor Gottfried Haberler, Cambridge, Massachusetts, U.S.A.
Mr. J. R. Hicks, Cambridge, England.
Professor Harold Hotelling, New York, New York, U.S.A.
Mr. John Maynard Keynes, Cambridge, England.
Dr. N. D. Kondratieff, Russia.
Dr. T. Koopmans, League of Nations, Geneva, Switzerland.
Professor Simon S. Kuznets, Philadelphia, Pennsylvania, U.S.A.
Professor Oskar Lange, Chicago, Illinois, U.S.A.
Professor Wassily Leontief, Cambridge, Massachusetts, U.S.A.
Dr. Jakob Marschak, Oxford, England.
Professor Frederick C. Mills, New York, New York, U.S.A.
Professor Wesley C. Mitchell, New York, New York, U.S.A.
Professor Henry L. Moore, Cornwall, New York, U.S.A.
Professor Giorgio Mortara, Rio de Janeiro, Brazil.
Professor Gunnar Myrdal, Stockholm, Sweden.
Professor Umberto Ricci, Giza, Egypt.
Dr. Charles F. Roos, New York, N.Y., U.S.A.
Professor René Roy, Paris, France.
M. Jacques Rueff, Paris, France.
Professor Erich Schneider, Aarhus, Denmark.
Professor Joseph A. Schumpeter, Cambridge, Massachusetts, U.S.A.
Dr. Hans Staehle, Cambridge, Massachusetts, U.S.A.
Lord Stamp, Shortlands, Kent, England.
Professor J. Tinbergen, The Hague, The Netherlands.
Professor Gerhard Tintner, Ames, Iowa, U.S.A.
Professor Felice Vinci, Bologna, Italy.
Dr. Abraham Wald, New York, New York, U.S.A.
Professor Edwin B. Wilson, Boston, Massachusetts, U.S.A.
Professor Theodore O. Yntema, Chicago, Illinois, U.S.A.
Professor F. Zeuthen, Copenhagen, Denmark.

THE ECONOMETRIC SOCIETY

Summary of Accounts, 1930–1939



Years ending Sept. 30	Expenses			Earned Income	Deficit
	Administration	Journal	Total		
1930–31	$ 405.44		$ 405.44	$ 27.00	$ 378.44
1931–32	1,069.86	$ 493.00	1,562.86	165.94	1,396.92
1932–33	1,486.50	4,456.26*	5,942.76*	1,774.19	4,168.57*
1933–34	1,508.82	5,054.98	6,563.80	2,866.68	3,697.12
1934–35	1,422.43	5,927.16	7,349.59	3,280.46	4,069.13
1935–36	1,809.32	4,972.27	6,781.59	4,083.73	2,697.86
1936–37	1,044.01	4,684.48	5,728.49	4,371.71	1,356.78
1937–38	919.60	4,764.34	5,683.94	5,151.67	532.27
1938–39	1,038.55	4,856.86	5,904.41	5,238.40	666.01

* Expenses for only the January, April, and July numbers of Econometrica are included in 1932–33 due to the fact that this was the first year of the Journal and the fiscal year ends September 30. If the October number had been added, the total expenses for the year would have been increased by approximately $700, to a figure about $100 in excess of the $6,653.80 reported for 1933–34.

The total deficit of $18,963.10 has been met in part by donations totalling $15,918.50. The remaining deficit of $3,044.60, together with a working balance of cash and other current net assets amounting to $39.70, is represented by notes of the Society outstanding to the amount of $3,000, plus current liabilities of $84.30. As security against this liability the Society, in addition to the cash balance, has a stock of approximately 375 complete files of Econometrica, Vols. 1–7, and about 4,400 extra copies of various issues.

Alfred Cowles
Treasurer

The University of Chicago

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The Committee on Scientific Aids to Learning, President Conant of Harvard, chairman, has made a grant to cover the cost of making a microfilm master negative, on the most expensive film, of sets of volumes of scientific and learned journals.

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ERRATA

Because of lack of time, the paper by K.-G. Hagstroem, "Remarks on the Theory of Depreciation," ECONOMETRICA, Vol. 7, October, 1939, pp. 289–303, was published without proof being read by the author, who now has discovered the following errors:

p. 292, line 23: *for* $\dfrac{\mu+\delta}{1-e^{-s(\mu-\delta)}}$ *read* $\dfrac{\mu+\delta}{1-e^{-s(\mu+\delta)}}$

line 27: *for* $\{P(\mu)-\mu\}\cdot s_{\overline{s-t}}$ *read* $\{P(\mu)-\mu\}\cdot s_{\overline{t}}$

p. 298, Table 2, 3rd column, 4th box from bottom: *for*

$$\left(\delta_0 \frac{k_0}{k}\ \delta\right) \quad \textit{read} \quad \left(\delta_0 - \frac{k_0}{k}\ \delta\right)$$

p. 302, line 6: *for* $\int_V^{\alpha} \lambda(z)dz$ *read* $\int_V^{\infty} \lambda(z)dz$

line 8: *for* $\int_z^{\alpha} \beta(z)dz$ *read* $\int_z^{\infty} \beta(z)dz$

for $\int_z^{\alpha} \lambda(z)dz$ *read* $\int_z^{\infty} \lambda(z)dz$